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XXXIII. Illustrative of Mr. A. Campbell's Paper on the Measurement of Wave Length for High Frequency Electrical Oscillations.

# PROCEEDINGS

OF

# THE PHYSICAL SOCIETY

# OF LONDON.

## I. *Observations on the Electric Arc.*

*By* WALTER L. UPSON, *E.E., M.S., Princeton*[*].

[Plates I. & II.]

ALTHOUGH numerous investigators have studied the phenomena of the electric arc in air, between carbon terminals, and also between metal terminals, yet the use of the electric arc in the production of electric oscillations raises some new questions with regard to it.

While I was working in the Pender Electrical Laboratory of University College, London, during the present session, Prof. J. A. Fleming, F.R.S., suggested therefore to me that it would be interesting to make a further examination of the arc between metal and carbon terminals, in air and hydrogen.

I am indebted to him for the facilities for carrying out the work, and also for many suggestions during its progress. The following apparatus was first constructed with the kind assistance of Prof. W. C. Clinton, B.Sc.

### DESCRIPTION OF APPARATUS.

Two upright brass tubes are connected across the top by a brass casting through which is bored a hole, to provide a means of causing cold water to circulate through a metal electrode.

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[*] Read June 14, 1907.

From the centre of the casting, extending downwards, is a cylindrical piece of copper, which ends in a tip about ½ inch diameter, on to which may be fitted the upper metallic terminal of the arc. The brass tubes are clamped to a slate base, and extend through it. Thus, by connecting the end of one of them to a water supply, a steady flow is obtained, and this is directed on to the very tip of the upper metallic terminal holder (see fig. 1, p. 3).

Through the centre of the slate base passes a screw of four inches' length, which is capable of being turned by means of a milled ebonite head, fixed at its lower end, and thus being made to approach or move away from the upper terminal holder. This screw likewise ends in a tip, by which the lower arc terminal may be held.

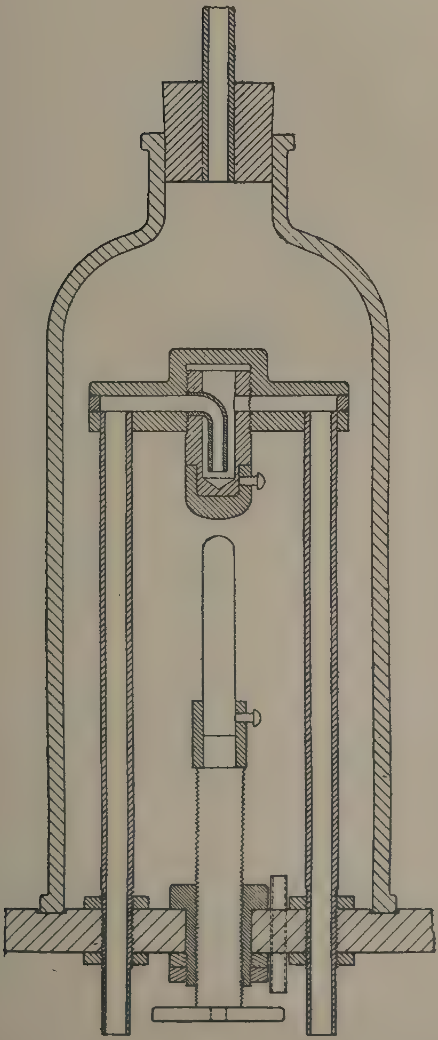
Surrounding the whole apparatus above the slate base was placed a bell jar, in the neck of which was a rubber cork holding a tube whose end was covered with wire gauze. There was also a tube let into the slate base, by which connexion could be made to a gas reservoir of any kind. The gas thus admitted was expelled through the tube fixed in the cork, and could, in the case of hydrogen or coal-gas, be made to burn at the end.

The two terminals of the arc were connected with the source of electric supply—usually the street mains at 110 volts, continuous current—the upper, through one of the tubes of the water circulation, and the lower through the central screw. Potential leads were connected to the other tube of the water circulation, and to the terminal leading to the central screw.

The electrodes used were ordinary solid carbons, 0·47, 0·4, and 0·37 inch in diameter, cored carbons 0·47 inch diameter; also metal terminals of copper, iron and aluminium, either as solid rods about 5 inches long, set into the lower holder, or pieces of the shape shown in fig. 2 (p. 4), fastened to the upper holder. The solid rods used were of the following diameters :—copper ⅜ in., iron ½ in., aluminium ½ in. and ⅝ in.

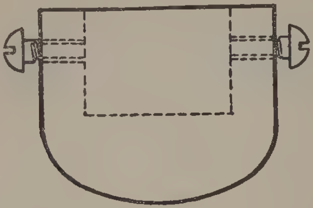
The light of the arc was passed through a double convex lens, and projected onto a sheet of squared paper, at such a distance that the arc length could easily be read off from

Fig. 1.



the image, in fractions of an inch. The arc was magnified to ten diameters. Its length could be kept constant by means of the screw holding the ower terminal.

Fig. 2.



When carbon terminals were used, the ends were previously shaped into the general form they would after use have assumed. By this means much time was saved, as it appeared that, with a given arc-length and current, terminals once shaped do not give variable volt readings according to the time the arc has burned, but may be considered in their normal state as soon as they have reached their normal temperature.

When metal terminals were used, the ends were rounded, but it was found better not to make them too pointed, owing to the tendency of the arc to bow out and become of uncertain and variable length.

## Description of Arcs.

In what follows, the arcs will be designated by the chemical symbols of their electrodes, followed by the gas in which they occur. Thus, Cu–C in H. The first electrode is always positive, and, except where carbon, or otherwise stated, is the upper terminal, cooled by water circulation. Observations were taken on arcs in air and hydrogen, using carbon, copper, iron, and aluminium in all combinations, both positive and negative; also several of them in coal-gas. The following is a summary of the observed phenomena.

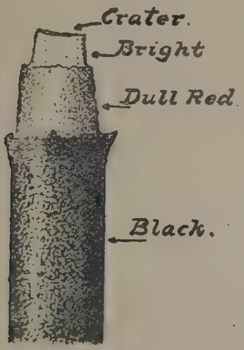
### *Arcs in Air.*

Cu–C and Fe–C arcs are probably more or less familiar.

Al–C. The characteristic aluminium flame is pale blue, being hardly distinguishable from that of iron. The Al was oxidized rapidly, but was only slightly eaten away. Very little deposit was left on the globe and framework. A strong odour was emitted, however, but a match applied at the end of the tube did not light the gas which should have been escaping. The arc could be lengthened to 0·5 in. with 10 amperes.

C–Cu. The arc was very steady; of a purple colour, except at the point of junction with the copper, where there was a small region of green colour. A small deposit of carbon quickly formed on the copper. If this were not continually removed, the arc soon assumed the characteristics of carbon–carbon. The deposit on the apparatus was very slight. The carbon terminal burned away quite rapidly, assuming a three-stage form, as shown in fig. 3.

Fig. 3.



C–Fe. This arc continually splutters and hisses, and is very hard to measure. It is very bright; more blue (from iron) than purple (from carbon). When the iron has reached a certain temperature, bubbles form—probably magnetite—and the energy consumption increases. The arc, however, now becomes steady. It was drawn out to 0·5 in. with 5 amperes and 80 volts across the arc.

C–Al. This arc started more readily than when its poles were reversed. There appears not to be any very heavy formation of aluminium oxide acting as an insulator on the end of the Al electrode. But once started, the arc was not very stable, and would not be drawn out to more than 0·3 in. with a current of from 10 to 12 amperes. The Al electrode was deeply pitted and very hard, containing a deposit of carbon embedded in it. The apparatus was coated with a light grey deposit of Al, but not so thickly as in the case of the Al–C arc in hydrogen. The end of the carbon was rounded, and showed indications of a crater form. A pungent odour was emitted as in the case of the same electrodes with current reversed.

Cu–Cu. A steady green arc could be drawn out to 0 2 in. with 2·5 amperes. The terminals were very little affected, being somewhat oxidized. A shallow crater was formed in the negative, and a small hard bubble of oxide on the positive.

Fe–Fe. This arc is very steady at low currents and small arc-lengths. It is blue-coloured, with a yellow aureole. As the arc-length and current are increased, yellow fumes are given off in considerable quantity, and a deposit of yellow-brown powder, no doubt iron oxide, covers the apparatus. The terminals, also, begin to boil and the arc becomes less steady. When the bubbles make their appearance the arc begins to hiss, and there appears to be an increase in the amount of power consumed in the arc. The boiling iron disturbs the arc, making it difficult to get accurate readings. There is also a hissing when the arc changes its point of attachment on the positive electrode. This is accompanied, as in the carbon-hissing arc, by a fall in voltage, but is quite distinct from the hiss which accompanies the boiling iron.

Al–Al. This arc was found to be very unstable. The terminals became quickly oxidized where the arc is formed, causing the latter to travel around continually, seeking unoxidized portions, and tending to lengthen out and run up the sides of the terminals, finally going out. The arc is a clear blue colour with a thin yellow aureole.

On each pole is a bright spot where the arc enters. A small grey deposit was formed on the framework.

Cu–Fe. The arc is bluish, with a yellow aureole. No copper-green is visible. The iron bubbles form a conducting oxide (magnetite), which, when cool, may be knocked off the end of the terminal. Also a bubble, or deposit of iron is left on the copper if the arc has been burned for a little time. There is also some brown iron-oxide deposit on the electrodes.

Fe–Cu. The copper was the upper cooled electrode. The arc showed a greenish colour, getting stronger at the copper electrode. Toward the iron it was blue. On knocking off the cap of oxide which covered the end of the iron, a heavy deposit of pure copper was found buried in the iron underneath. Quite a deep crater was burnt into the copper, while in the reverse experiment practically no effect was to be observed on it. The whole copper terminal was covered with a brown sooty deposit.

Cu–Al. The aluminium was the upper cooled electrode. This arc is more persistent than with the electrodes reversed in polarity. It is easily started, but is very unsettled in character, emitting constantly a hissing and spitting sound, and travelling about, over the surface of the electrodes, tending to lengthen out. But it does not go out as easily as the arc between the same terminals when the current is reversed. The flame is principally green from the copper. Sometimes the whole crater will be green with merely a thin aureole of blue. The aluminium electrode was covered with a golden-brown deposit from the copper, and its end was hard and pitted.

Al–Cu. The arc was blue in colour from the aluminium terminal, and green from the copper terminal, about equally divided. It was very unstable, and acted much like the Al–Al arc, being impeded in starting by the aluminium oxide. The arc would continually lengthen out until it ruptured itielf.

Fe–Al. The aluminium was above and cooled. The arc is as to be expected, blue with a yellow aureole. It shows the characteristic aluminium tendency to glide about and play around the edges of the electrodes. It is less steady than the copper arcs. The poles presented much the same appearance as those of the Fe–Cu arc. The Al electrode was embedded with iron to a considerable depth.

Al–Fe. The aluminium was eaten into very slightly. The iron appeared as in the case of the Cu–Fe arc.

### *Arcs in Hydrogen.*

C–C. The arc was pale blue in colour, with a faint purple core. Its length was difficult to measure on account of heavy carbon deposits. The carbons became shaped so that their ends formed parallel spherical surfaces, the positive being concave. Around the edge of the negative was built up a branching deposit of carbon, extending outward about a quarter of an inch. Compare arcs of carbon and iron.

Cu–C. The arc was small and pale, consisting of a central core of reddish-purple colour, enveloped by a region of green light. It persistently travelled around, finding no point of permanent attachment, and wearing a broad crater in the positive and shaping the negative to fit the crater. The end of the carbon became very hard, with a deposit of copper, while around the outside edge was soft carbon which easily fell away. The surrounding globe was blackened by a fine carbon deposit.

Fe–C. The arc is blue in colour with a reddish-purple core. The iron boils at its point of contact with the arc, its surface becoming irregular. A black deposit covered the apparatus. A kind of crater was eaten into the iron as in other metal positives, but was irregular, owing to the boiling which had taken place. The end of the iron was found to be so hard that a file would not cut it.

Al–C. This arc was more unstable than the other metal-carbon arcs. The globe and apparatus were heavily coated with a grey deposit of aluminium. The arc length was difficult to measure owing to the deep crater formed, and the shifting about of the arc. The aluminium wore away rapidly, but did not boil. It looked the same as in air, but no non-conducting formation acted to prevent the re-establishment of the arc. The arc was greyish-blue in colour, with a rather faint purple core.

C–Cu. This arc appeared brighter than other arcs in hydrogen. The carbon was eaten away very rapidly, but there was no apparent deposit of material in the globe. A grey carbon deposit on the copper terminal tended to make the arc one between carbon electrodes. The arc was more stable than when the current was reversed. It did not travel round on the copper surface and was fairly steady on the carbon except as it ate the latter away. There was no positive crater formed, the end of the carbon being quite irregular.

C–Fe. There was an intermittent discharge of sparks from all over the end of the carbon, and no steady arc. The discharge could not be maintained for any length of time by the currents used, but would constantly go out. A carbon deposit gradually formed on the iron, and black cobwebs became suspended from the framework and globe. (See fig. 4, Pl. I.) On continuing to send the current, it became possible to form a steady arc, but it was evident that we now had a carbon–carbon arc. On the carbon electrode there was built up a deposit of feathery branches which extended outward horizontally for $\frac{3}{8}$ inch all round. The iron was also heavily coated but not with branching deposits. The centre of the carbon was quite deeply cratered, the interior being light grey in colour and probably converted into graphite by the heat.

C–Al. This arc was extremely unstable. It would not continue at less than seven amperes. A small lump of carbon deposit appeared immediately on the Al electrode, and the carbon electrode burned away rapidly, the arc always remaining attached to the carbon deposit on the aluminium. There appears to have been no increase in the white aluminium deposit on the globe after the arc had been changed from Al–C to C–Al. This may have been because the arc was not attached directly to the aluminium surface.

Cu–Cu. With clean terminals no arc could be maintained in hydrogen with 110 volts supply and 15 amperes at contact, not even the shortest. On drawing the electrodes apart, a shower of sparks, or arcs, appeared, but immediately went out. Previously an arc was obtained, using a copper rod on which some carbon had been left from a former experiment. The arc remained until the carbon was exhausted.

Fe–Fe. No permanent arc could be maintained. The spark shower lasted only a little longer than that with copper electrodes, but no length at all appreciable could be obtained between the electrodes while the sparking lasted. The sparks were numerous and gave the same hissing sound that usually accompanies a spark discharge in a vessel. The terminals were reversed, but with the same result. The sparks were pale blue in colour. The lower iron rod presented the same appearance as in the iron–copper arc below. The upper electrode was covered with a black deposit, but at one spot was a bare globule of metal where the contact had been made.

Al–Al. This gave better results than Fe–Fe, or than Al in combination with either Cu or Fe. The arc was reddish-purple or magenta coloured, and travelled about continuously, finding no point of permanent attachment. The metal was eaten into somewhat as in other arcs of Al in H. Attempts were made to steady the arc by shaping the lower electrode. The worst effect was obtained when this was pointed, for the arc stretched out to such lengths as to continually rupture itself. The current was also far more variable, in this case, than when the ends of the electrodes were flat and parallel. The colour of the arc was found to vary with its length, being reddish for short arcs and more bluish as the arc-length increased. There was considerable dull grey deposit from this arc, and the upper electrode seemed to be considerably built up with aluminium carried over from the lower.

Cu–Fe. This was very unstable. It consisted of a shower of sparks which was continually going out. There was some black deposit on the globe.

Fe–Cu. This was very little different from the last. The iron electrode showed a surface of pure metal surrounded by the black deposit which seems characteristic of iron arcs in hydrogen. It was only slightly eaten away, but in pits of some depth. Some slight traces of copper were present on the end of the iron. The copper electrode was covered with the black iron deposit. At one point only, a small drop of pure metal like iron showed where the arc had made contact. On removing the deposit quite an amount of iron was found to have been carried over onto the copper. Thus we have, as with most of the arcs, evidence of the double transfer of material.

Cu–Al. The arc was a mixture of green and yellowish-red colours, and was very unstable. There was no particular deposit on the apparatus. The electrodes became roughened up somewhat, and the aluminium was eaten into considerably. The Al electrode was the upper cooled terminal.

Al–Cu. This arc was about as stable and bright as the last. The electrodes were burned away considerably, and the globe was much blackened. On filing the end of the copper, the metal looked like brass in streaks. The aluminium was covered with a slightly reddish-black deposit, and its end was found, on filing, to be porous to a considerable depth.

Fe–Al. This is a better arc than Al–Fe, being more in one place than a general spark discharge over the surfaces of the electrodes. It continually hissed, as is general with the metal arcs in hydrogen, but there was no accompanying voltage drop. There seemed to be a general black deposit. On the end of the iron was a metallic globule. The Al electrode was above, and cooled.

Al–Fe. A rotating mirror revealed a broad band of distinct sparks. The direct appearance was as though a number of arcs were continually jumping across the gap; in fact it resembled a medium between the ordinary metal arc and the momentary rain of sparks from Cu–Cu in H.

The sparks were bluish in colour, accompanied by heat-glow from the metal. The iron appeared little affected but the aluminium boiled considerably, and on making contact at start the metals tended to fuse together. A kind of crater formed in the aluminium. There was quite a heavy black deposit on the apparatus.

### *Arcs in Coal Gas.*

Cu–C. The arc was maintained with much more difficulty than in air, and required larger currents. Great quantities of carbon were deposited on the copper so quickly that it was impossible to measure accurately the arc-length. The arc was small and dull in appearance.

Cu–Cu. At about 28 volts and 8 amperes with an arc-length of approximately 0·01 in., a discharge took place, which in a moment went out. It could not be made continuous even with the smallest gap between the electrodes. On continuing the discharge for ten or fifteen minutes the arc suddenly started and remained steady to a length of 0·05 in. The current dropped considerably below 8 amperes, the arc continuing. The current was raised to 11 amperes and the arc burned quietly at 0·1 in. length, while a heavy deposit formed on both poles much as in the C–C arc in H. What really happens seems to be this:—With clean copper poles no arc will exist at currents up to 15 amperes and higher, as in hydrogen. But with a continued spark discharge carbon is separated from the coal-gas and deposited upon the poles. When there is sufficient carbon, the arc starts, and we really have a carbon–carbon arc. There was some copper deposit mixed with the carbon on the electrodes, but the latter were only very slightly burnt by the arc.

In general, it may be inferred that coal-gas is unsuitable for investigation on the actions of arcs between metal electrodes, for in a moment the carbon deposit makes the arc virtually one between carbon electrodes. The arc itself resembles those in hydrogen in brilliancy.

### *Maximum Arc Lengths.*

When the arc took the form of a spark-discharge, as with various metal–metal arcs in hydrogen, a maximum possible length of 0·05 in. was reached for all such arcs, provided both terminals were not of the same metal. This limit of 0·05 in. was with 110 volts supply and current up to 15 amperes. When the terminals are of the same metal the results vary. Copper and iron give no definite length of arc, and aluminium reaches its limit at 0·05 in. When carbon is one of the electrodes we have always a continuous arc. If carbon is negative there is a constant limit of 0·07 in. in hydrogen. But if the arcs are carbon–metal, the limit varies as follows:—

| Arcs in H. | Approximate limit of length (at 110 volts supply and 15 amp.). |
|---|---|
| Carbon–Iron................ | 0·05 inch. |
| Carbon–Aluminium .......... | 0·07 „ |
| Carbon–Copper.............. | 0·13 „ |
| Carbon–Carbon ............ | 0·07 „ |

In coal-gas, Cu–C gave an arc-length of 0·1 in. at 15 amperes.

In air, the arcs may always be drawn out longer than in hydrogen or coal-gas.

Cu–Cu gave 0·2 in. arc-length with only 2·5 amperes, which cannot be obtained from carbons with 110 volts supply.

The lowest limits found were—

for C–Al, 0·3 in. at 10 to 12 amperes,
and C–Cu, 0·26 to 0·34 in. „ 5 „ 6 „

The following comparisons of maximum arc-length were made, showing the effect of reversing the polarity when carbon formed one of the electrodes:—

| Arc. | Max. length. inch. | Current. | Voltage across Arc. |
|---|---|---|---|
| Al–C .......... | 0·5 | 10 | — |
| C–Al .......... | 0·3 | 10 to 12 | — |
| Cu–C .......... | 0·6 to 0·8 | 7 „ 9·5 | 70 to 53 |
| C–Cu .......... | 0·26 „ 0·34 | 5 „ 6 | 85 „ 75 |

*Note.*—Cu–C with a length of 0·3 in. at 6 amperes gave only 47 volts across the arc.

### *Volt-Ampere Characteristic Curves of Arcs* 0·05 *in. in Length.*

The length of arc, 0·05 in., was chosen as it was the longest which could be steadily maintained for all the arcs under examination. This length is so short that accurate measurements of unstable arcs are extremely difficult, and in such cases it is only fair to consider the curves given as generally approximate. Fig. 5 (Pl. II.) gives the volt-ampere characteristics of the arcs in air in which carbon was one of the electrodes. It will be seen that when the carbon is negative, it makes very little difference what the positive is in regard to the general position of the curve, while if carbon is positive a great variation occurs according to the material of the negative. The arc in which carbon forms both positive and negative partakes of the properties of both of these groups, occupying its individual position under the influence of the carbon negative; but being little influenced by its positive, it keeps the same general shape as the lower group of curves.

Fig. 6 gives this same group of curves for arcs in hydrogen. The metal–carbon and carbon–carbon arcs so nearly coincide that one curve is given to represent them. The real positions of this group may be seen in fig. 10. We now have not a difference in curvature, for the curves are approximately parallel, but merely variations in the distance apart, according to the material of the negative. The position of the curve, therefore, seems to be due to some constant of the material—say, its ability to become ionized in the particular gas.

In air there is superposed the active effect of oxidation. We may infer that the general shapes of the curves,—or, at any rate, variations in shape—are due to the influence of the surrounding gas.

Fig. 7 shows the characteristics of arcs in air when both electrodes are of the same metal. That of aluminium differs considerably from the other two, and in general it was found that where aluminium entered, either as one or both of the electrodes, there was apt to be considerable variation, whereas iron and copper kept pretty close together.

Fig. 8 gives the combinations of the different metals in air. Fig. 9 gives them for arcs in hydrogen. It has already been mentioned that these arcs bear more resemblance to heavy spark discharges than to genuine arcs. But the action seems to follow the same general law, in both cases. Some interesting comparisons may be made from these curves and those of figs. 5 and 6.

In figs. 8 and 9 we find the highest and steepest curves to be Cu–Al, and C–Al occupies a similar position in fig. 5. In figs. 8 and 9, again, the lowest and flattest curves are for Al–Fe. The copper and iron curves bear a general resemblance to each other throughout, but with a certain peculiarity.

We have considered the relative height of the curves to be due, in part, to the effect of the surrounding gas, and in part to a specific property of the material of the negative electrode. There may be other influences, but these two are suggested by a comparison of the copper and iron curves in figs. 5, 6, 8, and 9. The curves both for Cu–Fe and C–Fe are below the curves for Fe–Cu and C–Cu when the arcs are in air. But they are above when the arcs are in hydrogen. With carbon for the negative, copper's place is seen to be above iron in both air and hydrogen.

It was desired to obtain a comparison of the slopes of the curves for C–C in air and metal–C and C–C in hydrogen, in order to show the change in voltage for a given change in current in the several cases. All experiments agreed in the result shown in fig. 10, namely, that the C–C curve in air cuts the other curves in the neighbourhood of 6·5 amperes and 47 volts, and is a much flatter curve. In fact, all the air-curves derived are flatter than those in hydrogen with the exception of that for C–Al. This arc is difficult to measure, but very careful observations led to the conclusion that the curve was, if anything, steeper than as shown in fig. 5.

Curves were also taken of the copper–copper arc in air under three conditions :—

(1) + above, and cooled by water-circulation.
(2) − above, and cooled by water-circulation.
(3) Arc between two rods of 0·375 in. diam., with + above, and no cooling.

The latter curve lies between the others, but is somewhat steeper, as shown in fig. 11.

### *Voltage Equations of different Arcs.*

In determining the equations of different arcs, the form of procedure given by Mrs. Ayrton has been followed. The characteristics have first been sketched in according to the readings obtained. From these curves were drawn up tables of volts, amperes, and watts. The watts were then plotted against amperes. This should give straight lines if the volt-ampere curves are satisfied by equations of the form

$$V_l = a + \frac{b}{A},$$

where $a$ and $b$ are constants, $l$ = length of arc, and A = current in amperes. In all cases, the approximation to a straight line was so close, that no doubt could be entertained that the deviation was due to error of observation.

From these watt-ampere curves new sets of tables were drawn up, giving volts, and from these derived volts and amperes the curves shown in the different figures have been drawn. The derived curves have in all cases been found to satisfy the original points, and may be regarded as more nearly the truth than the original curves. From this we conclude that, provided the electrodes are of solid materials, all arcs of constant length are satisfied by the equation

$$V = a + \frac{b}{A}.$$

It is difficult to obtain long enough arcs to prove the validity of a like assumption with regard to arcs of constant current but variable length, which shall cover the whole range of arcs in air and hydrogen. But no case has been found where the arc of constant current is not represented by the equation

$$V_A = a + bl.$$

Equations of the general form for variable length and current given by Mrs. Ayrton,

$$V = a + bl + \frac{c + dl}{A}$$

have been derived in a number of cases, as follows.

For carbon–carbon in air, both solid, positive above, 0 475 in. diam.; negative 0·4 in. diam.

$$V = 38 + 1{\cdot}8\,l + \frac{19 + 12{\cdot}6\,l}{A},$$

where $l$ is given in millimetres.

For copper–carbon in air, electrodes as described,

$$V = 14{\cdot}6 + 2{\cdot}4\,l + \frac{33{\cdot}6 + 7\,l}{A}.$$

For iron–carbon in air,

$$V = 12{\cdot}8 + 2\,l + \frac{24{\cdot}3 + l}{A}.$$

For iron–iron in air, the positive cooled,

$$V = 18{\cdot}5 + 3{\cdot}97\,l + \frac{18{\cdot}5 + 3{\cdot}97\,l}{A}.$$

For copper–copper in air, the positive cooled,

$$V = 17{\cdot}4 + 3{\cdot}6\,l + \frac{18{\cdot}3 + 10{\cdot}8\,l}{A}.$$

For copper–copper in air, the negative cooled,

$$V = 22{\cdot}2 + 1{\cdot}5\,l + \frac{31{\cdot}1 + 5{\cdot}2\,l}{A}.$$

For copper–copper in air, both solid rods 0·37 in. diam., the positive above,

$$V = 18 + 17{\cdot}5\,l + \frac{52{\cdot}1 + 3{\cdot}43\,l}{A}.$$

For copper–carbon in hydrogen,

$$V = 21{\cdot}5 + 3{\cdot}46\,l + \frac{4{\cdot}3 + 86{\cdot}4\,l}{A}.$$

For aluminium–carbon in hydrogen,

$$V = 8{\cdot}45 + 7{\cdot}1\,l + \frac{84{\cdot}5 + 85\,l}{A}.$$

### *Alternating-Current Arcs.*

The carbon–carbon arc in hydrogen at about 50 volts, and 15 amperes at a frequency of 80, was steady to a length of 0·03 in. On first making contact, the carbons being cold, the arc gave a momentary flash which was repeated at each contact. By maintaining a bad contact for a few seconds, until the carbons began to glow, and then slowly separating them, the arc could be obtained, and was quite steady. It could not be drawn out longer than 0·03 in. with 100 volts supply.

Copper–carbon in hydrogen gave no arc at 100 volts, 15 amperes and 80 cycles, nor again at 100 volts, 20 amperes and 50 cycles. The arc apparently started at contact, but went out at the first zero of the current after separation of the electrodes.

Carbon–carbon in coal-gas gave an arc of 0·01 in. and 0·02 in., the volts and amperes being about 38 and 14 respectively, and the frequency 80. At times the arc seemed to reach 0·03 in. in length, but it did not appear to draw out as long as the same arc in hydrogen. As in hydrogen, the arc required a certain amount of previous heating of the carbons to start.

Copper–carbon in coal-gas gave at first no arc with 120 volts supply, and current of 15 amperes at 80 cycles. Sparking and heating, however, gradually produced a deposit of carbon on the copper, and when this was sufficiently thick, it was possible to maintain an arc of exceedingly short length.

Copper–carbon in air. The arc would not start until the carbon had been heated to a glow. By this time a deposit of carbon was on the copper. The arc could then be drawn out to 0·05 in., but with some difficulty, and with not less than 15 amperes. At 7 or 8 amperes the arc would hardly form at all.

Copper–copper in air would not start at all. It seems probable that if sufficient heat were concentrated at the ends of the poles, the arc could be maintained.

### *Interrupted Arcs.*

It was desired to compare the lengths of time that various arcs could be interrupted without losing the power of restarting themselves when the voltage was again applied to their terminals. For this purpose, apparatus was devised in accordance with a suggestion by Dr. Fleming as follows :—

A small electromagnet was fixed so that its armature would have a free fall when the current was switched off. When the armature was held by the magnet, two springs bore lightly against its sides and, they being in the main arc circuit, the current of the arc passed through the armature. When the armature was allowed to fall, it was guided into another set of spring-contacts in parallel with the first set, thus completing the circuit, which in falling, it had opened.

The length of fall could be measured accurately, and the time was calculated from the formula $t=\frac{\sqrt{d}}{4}$, where $d$ is the fall in feet.

It was especially desired to compare the carbon–carbon arc in air and the copper–carbon arc in hydrogen. With the currents used, the spark on opening the circuit was in all cases so small as to be of inappreciable length, and therefore its effect on the value of $t$ could be neglected. The following table is for solid carbons in air, with arc-length of 0·05 in.

| Volts. | Amperes. | Fall in feet. | Time (sec.). | Watts. |
|---|---|---|---|---|
| 45 | 4·5 | 0·3 | 0·137 | 202·5 |
| 49 | 3 | 0·146 | 0·095 | 147 |
| 43 | 6 | 0·541 | 0·184 | 258 |
| 41 | 7·5 | 0·833 | 0·228 | 307·5 |
| 40 | 9 | 1·44 | 0·3 | 360 |

Plotting either amperes or watts against time gives nearly a straight line (see fig. 12, Pl. II.).

Both lines, continued downward, cut through the origin. If, instead of curving downward at their upper ends, they are more properly straight throughout, we have

$$A=at, \quad \text{and} \quad W=bt,$$

where $A$=amperes, $W$=watts, $t$=time, and $a$ and $b$ are constants.

Using cored carbons, the arc was found to restart much more easily. In fact it was difficult to make the current so small that the arc would not restart after a time lapse of

0·3 sec. Observations were taken with constant time of interruption to find the effect of variations of arc-length on the current required to restart. The carbons were cored, each 0·47 in. diam.

| Volts. | Amperes. | Arc Length (in inches). | Time (in sec.). | Watts. |
|---|---|---|---|---|
| 47 | 1·2 | 0·02 | 0·3 | 56·4 |
| 45 | 2·5 | 0·05 | 0·3 | 112·5 |
| 49 | 3·7 | 0·1 | 0·3 | 181 |
| 55 | 5 | 0·15 | 0·3 | 275 |
| 50 | 7 | 0·2 | 0·3 | 350 |
| 55 | 8·5 | 0·25 | 0·3 | 467·5 |

Plotting either amperes or watts against arc-length gives practically a straight line (see fig. 13).

A cored carbon was bored out to a depth of about 1·5 inches, and packed with powdered potassium sulphate. With this as positive and an ordinary cored carbon as negative, a very low-power arc was obtained. The arc was perfectly steady at 20 volts, 1·1 amperes and 0·05 in. length. It was much more difficult to make this arc restart than the preceding. The results agreed fairly well, however, when the amount of energy consumed was the same.

The following observations were taken with this arc.

| Volts. | Amperes. | Arc Length (in inches). | Fall (in feet). | Time (in sec.). | Watts. |
|---|---|---|---|---|---|
| 12·5 | 5 | 0·05 | 0·125 | 0·089 | 62·5 |
| 12 | 6 | 0·05 | 0·416 | 0 162 | 72 |
| 12 | 8 | 0·05 | 0·916 | 0·239 | 90 |
| 11 | 10 | 0·05 | 1·44 | 0 3 | 120 |
| 12 | 12 | 0·1 | 1·44 | 0·3 | 144 |

Fig. 14 shows the variation of amperes and watts for both fall and time. In the curve of amperes against time there appears to be a lower limit of about 4 amperes under which the arc of 0·05 in. length will not restart. This accords with experiment.

A copper–carbon (solid) arc in air would not restart with an interruption of 0·072 sec., when the readings for the arc were 22 volts, 12·2 amperes, and 0·03 in. arc-length.

A copper–carbon (cored with $K_2SO_4$) arc in air restarted as follows :—

| Volts. | Amperes. | Arc Length (inches). | Fall (feet). | Time (seconds). |
|---|---|---|---|---|
| 12·5 | 13 | 0·05 | 0·083 | 0·072 |
| 12·5 | 12·5 | 0·03 | 0·125 | 0·088 |

A carbon(cored with $K_2SO_4$)–copper arc in air would not restart with an interruption of 0·072 sec., when the readings were 17 volts, 13 amperes, and 0·05 in. length.

A copper–carbon (solid) arc in air could not be made to restart with the smallest obtainable interruption, 0·051 sec., when its readings were 19 volts, 12·5 amperes, and 0·02 in. length. It did restart, however, when the arc-length was reduced to about 0·01 in.

A copper–carbon (solid) arc in coal-gas could not be made to restart at 23 volts, 14 amperes, and 0·01 in. length, in the shortest obtainable time of interruption. Even after running it some time, to obtain a good deposit of carbon on the copper, it would not restart.

A copper–carbon arc in hydrogen could not be made to restart at all. Its readings were as follows :—

| Volts. | Amperes. | Arc Length (inches). | Time of Interruption (seconds). |
|---|---|---|---|
| 32 | 12 | 0·02 | 0·056 |
| 24 | 13 | 0·01 | 0·051 |
| 23 | 14 | 0·005 (approx.) | 0·04 |

In conclusion, the author wishes to express his indebtedness to Dr. J. A. Fleming, F.R.S., and to Mr. W. C. Clinton, B.Sc., for their kind assistance during the progress of this work.

## Discussion.

Mr. W. Duddell said the paper was a most interesting one. He asked if the author had noted the character of the different arcs which he had used. The characteristic curves for arcs in coal-gas or hydrogen were steeper than those in air; and therefore arcs in coal-gas were more suitable for the production of oscillations. He asked if the curves remained steep when the current was quickly changed. He had noticed the phenomenon of an arc between two cored carbons relighting, and said that on high-voltage circuits the length of time was considerable and amounted in some cases to seconds.

Mrs. Ayrton said the paper was a valuable one, but she was sorry the author had not given a careful description of the appearance of the arcs and carbons under different conditions, and also the observations upon which the curves were founded. It was difficult to arrive at any general conclusions from the author's curves. The steep parts depend not only on the composition of the electrodes but on their size and on the length of the arc. The author had only used short arcs which with carbons only gave the flat part of the curve. With metal-carbon you get the steep part. Mrs. Ayrton also expressed her interest in the relighting experiments shown by the author.

Mr. A. Campbell asked if the characteristic curve of a mercury arc was similar to that of the other metal arcs investigated. He pointed out that the use of unscientific units (inches and feet) in giving the results of such interesting researches was most deplorable. In all standard work on the subject the lengths of arc are always given in millimetres, and even in commerce the diameters of carbons are similarly expressed (*e. g.* 12 mms. and not 0·47 inch).

Mr. Alexander Russell stated that Mr. Upson's paper would be extremely useful to all engaged in research in connexion with the electric arc. He agreed with the author that coal-gas was very unsuitable for experiments with the arc. He had recently been experimenting with high-voltage alternating arcs in coal-gas, and in most cases the enclosing vessel was soon filled with black cobwebs. When, however, the high-voltage high-frequency arc between iron-iron electrodes was shunted by a Leyden jar in series with a small coil of wire the discharge between the electrodes became simply a torrent of sparks and no appreciable quantity of soot was formed. The difficulty the author experienced in certain cases in obtaining an arc was due probably to the low voltages used. When high voltages are applied it is easy enough to maintain an arc. More experimental research was wanted on the upper part of the statical characteristic curve of the arc. A good deal of exploration has already been made in this direction in Dr. Simon's Laboratory at Göttingen with the help of a motor generator from which three or four amperes of direct current can be obtained at 5000 volts. The results already obtained were of great importance. The success of the statical characteristic method as an instrument of research was mainly due to the care with which Mrs. Ayrton had specified the method of obtaining the curve. He hoped that Mr. Upson would make further researches at high voltages.

Dr. Fleming congratulated Mr. Upson on his paper, and said that it represented a considerable amount of careful work, and even if all the conclusions to be drawn from it were not yet absolutely clear, it would furnish a statement of facts which would assist in future theorising. He said that the interesting part to him at the moment was the difference between the form of the volt-ampere characteristics for the carbon-carbon arc in air and the metal-carbon arc in hydrogen, but on that matter he would defer his remarks until he had shown some other experiments.

Mr. Upson, in reply, stated that the primary object of the paper was to compare metal arcs with carbon arcs. He had also performed some experiments with charges of voltage and variations in the length of burning of the arc, and he hoped to carry out a series of experiments using high voltages.

## II. *Some Observations on the Poulsen Arc as a means of obtaining Continuous Electrical Oscillations. By* J. A. Fleming, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London*[*].

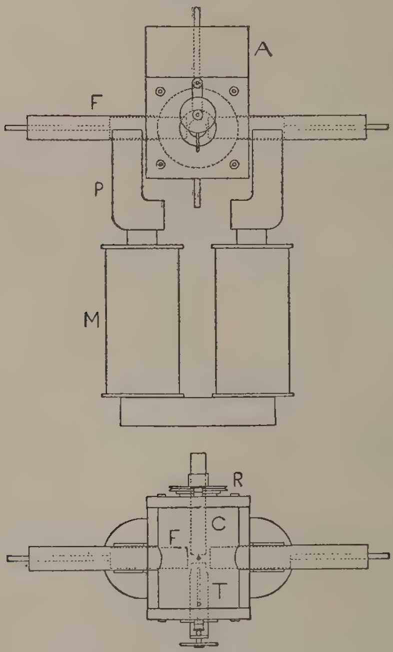
The interesting discovery of Mr. Poulsen that an electric arc between a carbon and a metal electrode formed in an atmosphere of coal-gas and traversed by a magnetic field, when shunted by a condenser and inductance, as in the Duddell musical arc, furnishes a means of obtaining undamped electrical oscillations, of much higher frequency than with a plain carbon arc in air, has attracted much attention from its technical application in wireless telegraphy. Having made a careful examination of the matter during the last six or seven months, the record of some of my observations in connexion with it may be of interest to other physicists who are studying the arc method of exciting oscillations. The apparatus with which I have worked, constructed on the same lines as that of Mr. Poulsen, consists of a stout brass cylinder 14 cms. in diameter and 19 cms. long, which is soldered across another brass box 7 inches by 18 cms. by 25 cms. high, the cylinder being open at both ends (see fig. 1). The interspace between the cylinder and the containing box serves as a water-jacket, and through it cold water is made

[*] Read June 14, 1907.

to flow by inlet and exit pipes. The ends of the cylinder are closed gas-tight by marble plates, and these are pierced by holes in which brass sleeve tubes are fitted.

The cylinder has also inlet and exit pipes for coal-gas. In addition, a pair of brass tubes pass through the sides of the box and the cylinder, and in these are fitted soft iron rods 4 cms. in diameter and 20 cms. long. The inner ends are about 3 to 4 cms. apart. In the chamber so constructed an electric arc is formed between the edge of a stout cylindrical carbon rod 2·5 cms. (1 inch) in diameter, which passes through the sleeve in one marble end. This carbon is kept in slow rotation by an electric motor geared to it so that it makes one rotation in about 2 or 3 minutes. In apposition to this carbon is a hollow brass tube closed at both ends, having cold water circulating through it. This passes through

Fig. 1.



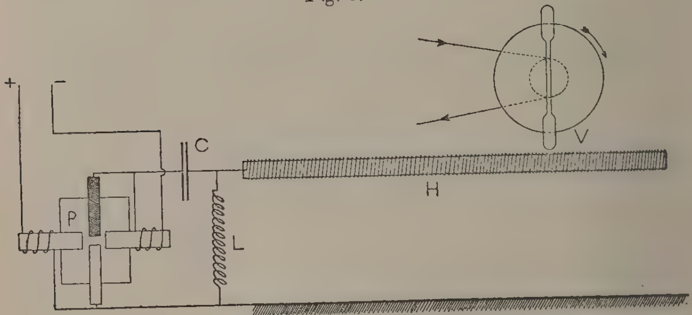
the other marble end, and it is terminated by a copper beak. The electric arc is formed between this beak and the sharp edge of the rotating carbon. The copper electrode can be moved to and fro by an adjusting screw, and is so set that the arc is formed in a magnetic field between the soft iron pole pieces. The magnetic field is created across the arc by setting the brass box upon an electro-magnet and joining its poles to the two iron cylinders by suitable soft iron connectors. With this apparatus it is possible to create an electric arc of suitable length in an atmosphere of coal-gas, the arc being also traversed by a magnetic field. The carbon is the negative and the copper beak the positive pole. The field used did not exceed 1000 C.G.S. units in strength, and generally was about 600 units. This field was so directed as to cause the arc to spring between the upper edge of the carbon and the copper beak, and be forced upwards in an arch.

Using a direct current supply at a pressure of 400–500 volts with a suitable rheostat interposed, it is easy to form an arc having a potential difference between the carbon and metal electrode of 300–350 volts and taking 5–10 amperes.

These electrodes are then connected by an oscillatory circuit consisting of a condenser made with metal plates separated by sheet ebonite, the whole being immersed in an insulating oil. The capacity of the condenser generally used by me was 0·00289 microfarad. In series with the condenser was an inductance consisting of a coil of insulated copper wire No. 7/22 wound in square form over a wooden cross so as to make a circuit 60 cms. in the side, having 8 turns. The inductance of this coil for high-frequency currents was 116,200 centimetres. An additional sliding inductance was used consisting of bare copper wire wound in a spiral on a wooden rod $1\frac{1}{4}$ inch or 3·2 cms. in diameter, having 8 turns to the inch. Two such spirals were placed on a support parallel to each other, and cross-connected by a sliding copper saddle, so that by moving the saddle, more or less inductance could be thrown into the circuit without interrupting it. The full inductance of the double spiral was 200,000 centimetres. If then an arc is created between the electrodes and adjusted to be of such a length as to take about 8 amperes with 300–350 volts between the carbons, then powerful oscillations are set up in the square circuit in series with the condenser. A hot wire ammeter inserted in this circuit indicated a current of about 5 amperes. If an electrostatic voltmeter is joined across the terminals of the condenser, we find a potential difference of 1200 to 1500 volts (R.M.S. value), vastly exceeding the steady potential difference of the arc electrodes.

The existence of a powerful oscillatory current in this square circuit may be shown by holding near to it a circular coil of about 30 turns, 30 cms. in diameter, formed of well-insulated wire, to the ends of which a 50-volt carbon filament glow-lamp is attached. When this round coil is held near and parallel to the square coil, the lamp lights up brilliantly in consequence of the secondary current induced in the lamp coil.

Fig. 2.



It is, however, seen that the lamp often flickers a good deal, showing that the primary oscillation current is not absolutely constant, and this flickering varies with the adjustment. It is, however, possible to adjust the arc so that the glow-lamp remains fairly steady, and if its incandescent filament is examined in a revolving mirror, its image will be seen spread out into a uniform band of light.

This, however, is not an absolute proof of the uniformity of the oscillatory current. To the above-mentioned condenser circuit, I then attached a long helix of fine silk-covered wire (No. 30 S.W.G.) wound in one layer of closely adjacent turns on an ebonite tube (see fig. 2). The length of this helix was 210 cms., its diameter 4·78 cms., and the number of turns 5470. This helix was supported horizontally on insulating stands about 45 cms. above the table, a sheet of zinc as long as the helix being placed on the table to act as an earth plate. This earth plate was connected to the metallic electrode of the arc and the helix to the junction point between the condenser and inductance coils in series with it.

The far end of the helix was provided with a number of needle-points formed by a bunch of sewing-needles attached to a wire.

On adjusting the inductance in series with the condenser so that the whole inductance in series across the arc terminals had a value of about 215,000 cms., powerful stationary oscillations were excited in the helix. Under these conditions, vivid blue electric brushes 1 or 2 cms. long appear on the needle-points at the far end of the helix, thus showing that the potential at that end has been increased by resonance to many thousand volts. These brushes are not of the same harmless character as the ordinary electric brush of an electrostatic machine, but are veritable flames or arcs taking place into the air. They are very hot and will melt sealing-wax and set fire to paper, and inflict the most painful burns if incautiously touched. They make a rushing or whistling sound and are of a strong blue colour.

If the inductance in the condenser circuit is varied even a little, the brushes disappear. The above described helix considered simply as an open circuit oscillator, has a natural time period of electrical oscillation of very nearly 5 one-millionths of a second, or a frequency of $0{\cdot}197 \times 10^6$. This value had been exactly determined in previous experiments with it*. As the value of the capacity in the condenser circuit is 0·0029 microfarad, and the total inductance in series with it 215,000 cms., the time period T of the condenser circuit is given by:

$$T = \sqrt{\frac{0{\cdot}0029 \times 215000}{5{\cdot}033 \times 10^6}} = \frac{5}{10^6} \text{ nearly;}$$

and this shows that to obtain the brush-discharge effect the condenser circuit has to be exactly tuned to the period of the fundamental electrical oscillation of the helix.

The effect of these oscillations in the wire is to create an intense electric field round the helix extending in all directions for a distance of nearly a metre. Vacuum-tubes of all kinds glow brilliantly when brought near the helix. Especially brilliant is the glow in the form of Neon tube I employ with my cymometer, which, as I showed some three years ago, is extremely sensitive to a high-frequency electric field. The form of tube I use is of the spectrum type, the straight part being a glass tube with a bore of 1 mm., and the enlarged ends cylinders of about 1 cm. diameter and 5 cms. long.

The beautiful discovery made some time ago by Sir James Dewar that Neon and other rare atmospheric gases can be separated from the commoner constituents of air by means of charcoal cooled to very low temperature, renders it possible to fill spectrum vacuum-tubes with nearly pure Neon, and such tubes are of very great utility in high-frequency research. When filled with Neon at the proper pressure they glow with an intensely brilliant orange-red light in a high-frequency field. If one of these tubes is held near the helix when in action, it shows by its glow that there is an electric field increasing in intensity all the way up the spiral to the free end.

But more, the Neon tube shows us that the oscillations are not perfectly continuous. If the tube is waved rapidly to and fro in front of the spiral, by the persistence of vision its image is expanded into a fan-shaped area of light. Examining this, we see it crossed vertically by narrow or broad lines irregularly placed. This indicates that the tube is extinguished at intervals, and this is because the electric field disappears and the oscillations are interrupted. Not only Neon tubes, but all other vacuum-tubes show this effect provided they are narrow tubes. If a wide tube is used, the effect is not seen because the intermittent images of the tube overlap each other. The phenomenon is well shown by rapidly rotating a Neon tube near the spiral. The tube must be fastened by silk threads to a strip of ebonite held on an

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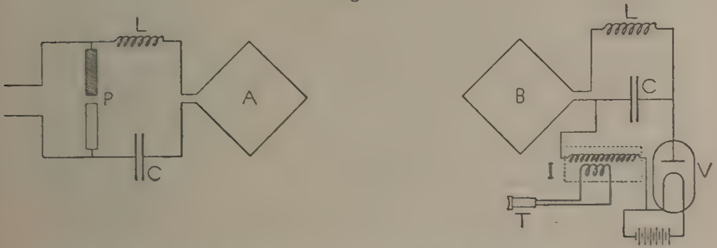
* See J. A. Fleming, "On the Propagation of Electric Waves along Spiral Wires," Phil. Mag. ser. 6, vol. viii. p. 434 (October 1904).

ebonite axle, and this rapidly rotated by a turn-table (see fig. 2).

The intermittency is always most marked when the arc is first started and then diminishes, but it is always present more or less, and it shows that the oscillations, even if undamped, are not absolutely continuous, but are cut up into irregular groups. It seems to be caused by the arc changing its point of departure on the carbon electrode. It is increased if the rotation of the carbon is irregular, but I have never been able to abolish it altogether. Another proof of the same fact is derived as follows:—

The helix is removed and another square coil of 12 turns or so of insulated wire wound on a frame 60 cms. inside is placed parallel and at a distance of 2 or 3 metres from the similar coil in the condenser circuit (see fig. 3). This secondary coil is tuned by a condenser to the same frequency as the arc condenser circuit. Across the terminals of the secondary circuit another circuit is joined which consists of the oscillation valve or glow-lamp wireless telegraph detector devised by me four years ago, and already described*. This valve stops all movement of electricity in one direction, and in series with the valve is placed a telephone. On listening to the telephone when the arc is in operation and is quite steady, a crackling sound is heard, which is not sufficiently regular to be called a musical note. If the oscillations in the arc circuit were continuous and absolutely persistent, they

Fig. 3.



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* See J. A. Fleming, "On the Conversion of Electric Oscillations into Continuous Currents by means of a Vacuum Valve," Proc. Roy. Soc. Lond. vol. lxxiv. p. 476 (1905). Also Phil. Mag. May 1906.

would induce similar secondary currents, and my oscillation valve would separate out all the unidirectional constituents, and send through the telephone a perfectly steady continuous current. But this is not the case. The telephone yields a sound which shows that the continuous current through it is interrupted irregularly, and this can only be because the oscillations in the arc circuit are interrupted. If the telephone is not inserted directly in circuit with the valve, but is inserted in the primary circuit of an induction-coil, or air-core transformer the secondary of which is inserted in series with the valve, the sound in the telephone is still more increased.

We have therefore strong indications that the arc method of exciting undamped oscillations gives rise to irregular groups of oscillations which are separated by short intervals of time.

In order to be successful at all in producing high frequency oscillations by the arc method, I find it essential to pay attention to a number of details. One important condition of success is, as Mr. Poulsen has already pointed out, that the carbon electrode should be kept in slow rotation. If the carbon is not rotated, a bulbous deposit of carbon soon grows on the electrode and builds up a beak or pimple of carbon which makes the arc irregular and soon puts it out. The end of the round carbon should be cut off square to begin with, but it soon gets rounded off and a little corrugated or milled, and seems to work better when it has been used for a short time. Hard arc carbons are better than soft for producing oscillations. Secondly, as regards the gas. Pure hydrogen works with difficulty. Coal-gas is better, but the coal-gas cannot be used over and over again, as it is altered in composition by the arc, and this seems to create a difficulty in the use of the Poulsen arc as a transmitter in wireless telegraphy on board ship. The deposit of soot in the chamber is sometimes excessive. The coal-gas must be passed through quickly, but not too quickly, and my experience is that if the sides of the containing chamber are not kept cool, the deposit of soot is greatly increased. The more enriched the coal-gas is the worse becomes this trouble. I find it necessary in any case to clean out the chamber after a few hours of continuous use.

In the next place, the magnetic field required careful adjustment. If it is too strong, the arc cannot be maintained without constant interruption. If too weak, the oscillations are feeble.

The various factors—gas, field, arc-length, rotation of carbon and water cooling—all require exact adjustment to obtain the best result. In laboratory or in lecture experiments, when effects are only required for a few minutes, this is not difficult; but if the apparatus is used as a transmitter in wireless telegraphy, it must be possible to maintain constant for hours together the production of the oscillations, and so far I have not found that this is a very easy thing to do. I have tried many methods for making the arc self-regulating as to length, but without success. The best plan is hand regulation by a skilled assistant. There is a particular length of arc which for given current and voltage produces the oscillations most effectively, and this length has to be maintained constant.

When employed as a wireless telegraph transmitter, the question which presents itself for consideration is the efficiency of the arrangement as an energy transformer. We supply the arc with continuous current power, and we create by it high-frequency persistent oscillations in the antenna. There are several sources of energy dissipation. In the first place, the arc must be arranged in series with certain regulating or ballast resistance to permit of adjustment. I have found that very good effects are obtained when working off a 400-volt public supply of direct current or from a 500-volt continuous current 5 k.w. dynamo. Then from 50 to 100 volts may be dropped in the regulating resistance when working with an arc of 8–10 amperes.

In the next place, there is a large production of heat in the chamber containing the arc, the greater part of which is removed by the cooling water. We have then a portion of the power converted into high-frequency current in the condenser circuit, and this dissipates itself partly as heat in the inductance and condenser, and only a fraction of this power is radiated by the antenna. We are not really concerned to know the manner in which these heat losses are distributed. We cannot subject the matter to mathematical discussion unless we make certain assumptions as to the form of the curve representing the oscillations, which form has not yet been determined. I have found by experiment that the power factor of the arc when working is not far from unity, and hence the power given to the arc in the form of continuous current can be very approximately determined by the product of the arc current and the supply voltage. For instance, in one case the potential difference of the arc electrodes was found to be 320 volts and the arc current 8·4 amperes, whilst in two separate measurements with the wattmeter the power taken up was found to be 2632 and 2674 watts. The product of amperes and volts is 2688, and hence the power factor by these measurements is 0·98 or 0·99, or practically unity.

If the current through the arc can be kept sufficiently constant, which is rather difficult, the power radiated may be also measured by the wattmeter. For suppose that we have a direct-coupled antenna joined at some point to the condenser circuit, we may then take wattmeter readings with and without the attached antenna, and the difference will be the power radiated.

In the case of the long helix above mentioned, the power given to the arc when the helix was attached to the condenser circuit was 2674 watts, as measured by the wattmeter, and with the helix removed it was 2632 watts, the arc current remaining constant at 8·4 amperes. Hence in this case it appears that the radiation and power consumption due to the attachment of the helix is 42 watts. The measurements, however, cannot be made very accurately in the case of small power radiation.

In the case of a direct-coupled antenna which is emitting a wave-length corresponding to its fundamental oscillation, the radiation can be approximately calculated from a formula given by Hertz. Hertz shows that in the case of a dumbbell oscillator having an electric moment Q, the energy E radiated per period is given by the expression

$$E = \frac{16\pi^4 Q^2}{3\lambda^3} \text{ ergs per period,}$$

where $\lambda$ is the wave-length of the radiation.

If C is the capacity of one-half of the oscillator with respect to the other, and V is the maximum potential difference of the parts during an oscillation, and $l$ the effective length, then $Q=CVl$.

If I is the maximum value of the current in the oscillator during a period, then

$$I = 2\pi n CV \quad \text{and} \quad Q^2 = I^2l^2/4\pi^2n^2,$$

where $n$ is the frequency.

Accordingly,

$$E = \frac{4\pi^2 I^2 l^2}{3\lambda^3 n^2} \text{ ergs per period.}$$

Now $$u = 3 \times 10^{10} \text{ cms./sec.} = n\lambda$$

is the velocity of radiation.

In the above equation, I is the maximum current in electrostatic measure. If A is the ampere value of the same current, then $\frac{A}{10} = \frac{I}{u}$. If, then, the oscillations are persistent and with frequency $n$, the energy in ergs radiated per second from the antenna is

$$W = \frac{4\pi^2 I^2 l^2}{3\lambda^2 n} \text{ ergs per second ;}$$

$$= \frac{4\pi^2 u}{300} \frac{l^2}{\lambda^2} A^2 \text{ ergs per second ;}$$

$$= 40\pi^2 \frac{l^2}{\lambda^2} A^2 \text{ watts.}$$

For any antenna the ratio of wave-length $\lambda$ emitted to effective length $l$ is a fixed quantity and approximates in practice for single-wire antennæ to 2·5. We have, therefore, this curious result which does not seem to have been previously noticed, that the power radiated in watts from a linear antenna of any length is given by the expression $W=64A^2$, where A is the maximum value of the current flowing into the base of the antenna. If (A) is the current as read on a hot-wire ammeter inserted at that point, then

$$(A) = \frac{A}{\sqrt{2}}, \quad \text{and} \quad W = 128(A)^2 \text{ watts.}$$

Thus, if such a hot-wire ammeter showed an antenna current of 1 ampere (R.M.S. value), the radiation would be 128 watts, and for the same current is quite independent of the height of antenna or the frequency.

Seeing, then, that open antennæ have such enormous radiative power, we can in practical radiotelegraphic work employ very small antenna currents when using persistent oscillations.

A complete explanation of all the phenomena involved in the Poulsen arc can hardly yet be said to have been reached. It has been shown by experiments made under the Author's direction in the Pender Electrical Laboratory of University College, London, by Mr. W. L. Upson, that the characteristic curve or volt-ampere curve for a carbon-metal arc in hydrogen is a curve falling down with a much steeper slope than the characteristic for a carbon-carbon arc in air.

Also that the time during which the electromotive force can be removed without permanently extinguishing the arc, is very much shorter in the first case than in the second. Both these facts undoubtedly contribute to enable the production of oscillations of a higher frequency to be obtained with a carbon-metal arc in coal-gas, as compared with those which can be reached with a carbon arc in air.

In order that oscillations may be produced when a condenser circuit is shunted across the arc electrodes, the condenser, during its charge and discharge, must cause a variation of current through the arc, and this in turn must cause a variation in the potential difference of the electrodes in such direction as to assist the charge or discharge. Thus when the condenser is robbing the arc of current and so decreasing the arc current, this must automatically raise the potential difference of the electrodes, and when the condenser is discharging it must decrease the potential difference. The greater this P.D. variation for a given current decrement or increment, the greater will be the energy conveyed to the condenser circuit.

In the case of a carbon-carbon arc worked at 10 amperes and say 60 volts arc P.D., the characteristic curve is rather flat. Hence, if we employ a small capacity in the shunt circuit we have very little variation of arc current and arc

P.D., and little or no energy in the condenser-circuit oscillations. We can, however, obtain more energy by using, as Mr. Duddell originally did, a condenser of rather large capacity, say 1 to 5 microfarads. Moreover, the carbon arc has a persistence which renders it irresponsive to very rapid variations of current. Hence with a large current carbon-carbon arc in air, we can only obtain rather slow oscillations.

If, however, we employ a carbon-metal arc, carbon negative and cooled copper, positive electrodes in hydrogen or coal-gas, we then have an arc with a very steep characteristic curve, and one which responds to exceedingly rapid variations of current through it. We can, therefore, employ in the shunt circuit a condenser of small capacity, and yet convey to it a considerable amount of energy owing to the large P.D. variation in the arc caused by a small arc-current variation. Hence we can produce oscillations of high frequency. This theory is confirmed by the observation that in the case of the carbon-carbon arc in air for small currents, and in the case of the carbon-aluminium arc, the characteristic curve is nearly as steep as for large current carbon-metal arcs in hydrogen; and with these particular arcs, viz., the small current carbon arc in air and carbon-aluminium arc in air, we can obtain oscillations of high frequency.

Much further investigation is, however, necessary to enable us to state precisely the action of the surrounding medium, hydrogen, or better coal-gas, in producing this steep characteristic volt-ampere curve for the carbon-metal arc in these gases.

## DISCUSSION.

Mr. W. DUDDELL expressed his interest in the experiments, especially those in which it was shown that the oscillations were broken up into groups. He pointed out, however, that it was possible to obtain continuous oscillations with a musical arc using high frequencies. The question as to the measurement of the power put into the Poulsen Arc was of great interest, as also was the question as to the amount of power which could be taken from the secondary without stopping the arc. He questioned Dr. Fleming's statement that the power put into the arc was equal to the product "AV," and said that the power should be measured with a wattmeter.

Mr. A. Campbell remarked that Mr. Altberg in Odessa had lately shown that sparks of high oscillation frequency produce actual air vibrations of the same frequency, and by measuring the wave-lengths of these air vibrations by diffraction methods he had been able to determine the high frequency.

Dr. Fleming, in reply, said he was prepared to accept Mr. Duddell's criticism that the power put into the Poulsen Arc should be measured with the wattmeter. The really important matter, however, was the reason why the metal-carbon arc in coal-gas could produce oscillations of so much higher frequency than the carbon arc in air. This he thought was intimately connected with the slope of the characteristic curve.

## III. *A Form of Cosine Flicker Photometer.*

### *By* J. S. Dow, *A.C.G.I.*, *B.Sc.*[*]

[Plate III.]

The illumination of the white surface employed in any photometer is equal to $\frac{I \cos\theta}{d^2}$, [where I equals the intensity of the source illuminating the surface, $d$ the distance of this source from the surface, and $\theta$ the angle between the rays of light striking the surface and a normal to the surface].

Hence, when measuring the intensity of a source of light, we may either vary "$d$," in which case we utilize the inverse square law, or $\theta$, in which case the cosine law is utilized.

While the inverse square law is almost invariably utilized in photometric measurements, this method is inconvenient in one respect. In order to vary "$d$" the photometer is usually moved to and fro between the two sources of light to be compared. The observer is therefore obliged to be continually moving his head in order to follow the motion of the photometer, and this is particularly distracting when the eye is applied to a telescope. In order to avoid this necessity, many workers prefer to keep the photometer stationary and to move one of the sources of light. But in the case of gas-lamps and many other sources of light, this method is obviously unsatisfactory, and, even in the case of glow-lamps, is sometimes inconvenient.

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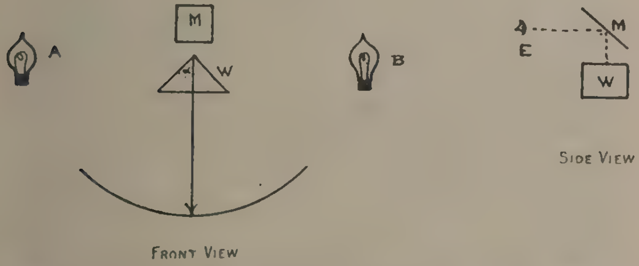
[*] Read June 28, 1907.

The utilization of the cosine law is advantageous in this respect, for the photometer may then be kept stationary and the illumination of the photometrical surfaces adjusted in the photometer itself. The type of instrument about to be described by the author, and shown in fig. 1 (Pl. III.), has this advantage. Indeed, while it is desirable that such a photometer should be mounted on a photometrical bench in the usual way for accurate work, the author has used the apparatus shown in the Plate for many purposes, such as the comparison of the results obtained by the methods of flicker and equality of brightness, without employing a photometrical bench at all. Also it is evident that the convenience of an instrument of this type is independent of the distance between the sources of light, provided the illumination of the photometrical surface is not too low.

The general principle of this instrument is shown in fig. 2.

Fig. 2.



The two sides of the rectangular Ritchie wedge, W, are illuminated by the two sources A, B. Above the wedge a 45° mirror, M, is placed so that the observer's eye at E sees an image of the illuminated surfaces reflected in this mirror. The wedge is arranged to rotate about the line of intersection of the photometrical surfaces as a horizontal axis. To the observer, therefore, this line appears stationary as the wedge is rotated.

Suppose now that the two sources A, B, are equidistant from the wedge.

Then, if the intensities of the two sources are the same, photometrical balance is obtained when the wedge is placed symmetrically about a vertical axis, as shown. If, however, A, say, is the brighter of the two sources, the wedge must be rotated in a clockwise direction, so that the rays from A strike the surface presented towards A more obliquely than before, while, conversely, the rays from B strike the surface presented less obliquely.

Hence let $\alpha$ represent the inclination of either surface to the vertical, when the wedge is in its symmetrical position.

Let $I_1$, $I_2$ represent the intensity of A and B respectively.

The illumination of the surface facing A is then $\frac{I_1 \cos \alpha}{d^2}$. Similarly, the illumination of the surface facing B is $\frac{I_2 \cos \alpha}{d^2}$; hence if $I_1 = I_2$ the surfaces appear equally illuminated and we obtain photometric balance, with the wedge in its symmetrical position.

Suppose now that $I_1$ is greater that $I_2$, and that it is therefore necessary to rotate the wedge through an angle $\theta$ in a clockwise direction, in order that the illuminated surfaces may appear equally bright.

The illumination of the surface presented to A is now $\frac{I_1 \cos(\alpha + \theta)}{d_2}$, and the illumination of the surface presented to B is $\frac{I_2 \cos(\alpha - \theta)}{d^2}$.

Therefore
$$\frac{I_1}{I_2} = \frac{\cos(\alpha - \theta)}{\cos(\alpha + \theta)}.$$

The ratio $\frac{I_1}{I_2}$ corresponding to each value of $\theta$ is therefore known. In this case the angle $\alpha$ was made 45° for, by so doing, the maximum angular range of photometrical reading is obtained.

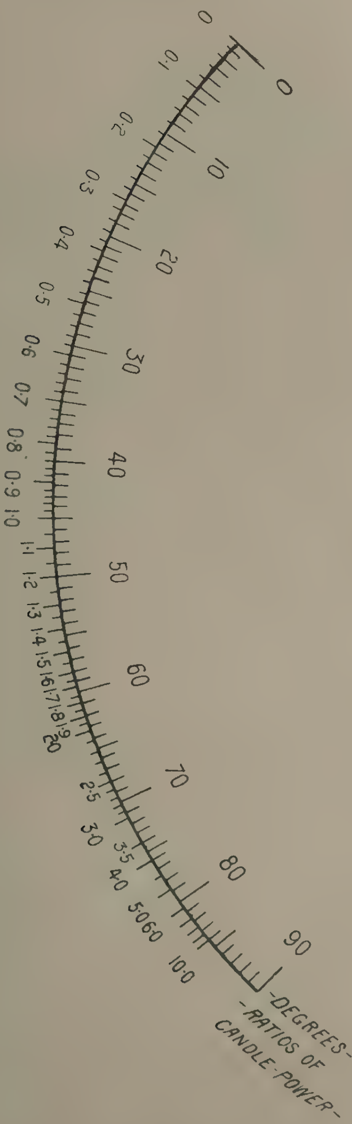

0
10
20
30
40
50
60
70
80
90
-DEGREES-
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.5
3.0
3.5
4.0
5.0
6.0
10.0
-RATIOS OF CANDLE-POWER-

Fig. 3.

A lever, rigidly attached to the wedge and utilized by the observer to rotate it, also served as a pointer indicating the value of $\theta$ and the corresponding ratio $\frac{I_1}{I_2}=\frac{\cos(45-\theta)}{\cos(45+\theta)}$, on a scale attached to the instrument.

This arrangement is shown in fig. 1.

The scale is also diagrammatically exhibited in fig. 3 (p. 39). The most serviceable portion of this scale lies in the neighbourhood of 0·2 to 5·0. Outside this range the values of $\frac{\cos(45-\theta)}{\cos(45+\theta)}$ change too rapidly with increasing values of $\theta$, for satisfactory readings. At the extreme ends of the scale, too, the rays necessarily strike one of the photometric surfaces very obliquely, and therefore emphasize any inequalities or roughness in texture of the surface and reduce the sensibility of the instrument.

At present fine plaster of paris has been used by the author to secure white matt surfaces. Under these conditions, the theoretical readings on the scale deduced from the cosine law agreed very closely with those obtained experimentally and assuming an inverse square law.

Between the limits 0·2 to 5 the agreement was within 3 per cent., and probably the substitution of perfectly ground photometric surfaces for those prepared by hand would result in still closer agreement.

Beyond these limits, however, the agreement was naturally less satisfactory. Hence it seems desirable, when using an instrument of this type to assign convenient distances to the sources of light and to utilize the most open, central, portion of the scale.

When using a photometer with inclined photometric surfaces such as occur in the Ritchie wedge, attention must be paid to the possibility of "angle-errors," *i.e.* errors introduced by uncertainty as to the exact angle at which the rays of light strike the illuminated surfaces. In this case a considerable rotation of the photometer, as a whole, about a vertical axis will alter the illumination of both surfaces equally and will, therefore, not affect the photometric result. But if the two sources are not in true alignment with the photometer, or if the photometer is tilted slightly about a horizontal axis, the illumination of the photometric surfaces may be differently influenced thereby, and errors may result.

Suppose, for instance, that we are comparing the intensities of two glow-lamps each half a metre distant from the photometer. Then, if the centre of illumination of one of these sources is raised say, one centimetre, the angle at which the rays from this source strike the surface presented to it is altered by about 0·6 degrees. This corresponds to an alteration of illumination of about one per cent., when the most open, central portion of the scale is used.

It is desirable, therefore, that the sources of light should be distant not less than, say, 1 metre from the photometer. It should then be possible to adjust the alignment of the lamps with sufficient exactitude to avoid appreciable error from this source. Indeed, in making accurate experiments with *any* photometer, it is desirable that the distance of the sources from the photometer should not be less than this value. Otherwise the inverse square law cannot be rigidly applied owing to the fact that the centres of illumination are rarely correctly located *.

A tilt of only one degree of the photometer, as a whole, about a horizontal axis would cause an error of about 4 per cent., even when the most favourable portion of the scale is used. It is, therefore, desirable that no appreciable tilt of this character should be introduced either by inequalities in the level of the bench or undue play in the carriage of the photometer.

Yet, it may be pointed out that the "double comparison method," now almost invariably used when careful photometric tests are attempted, eliminates all want of photometric symmetry in a stationary photometer and would completely eliminate the above source of error, if the same portion of the scale was used in both experiments.

This instrument can be utilized either on the Equality of Brightness or Flicker principle.

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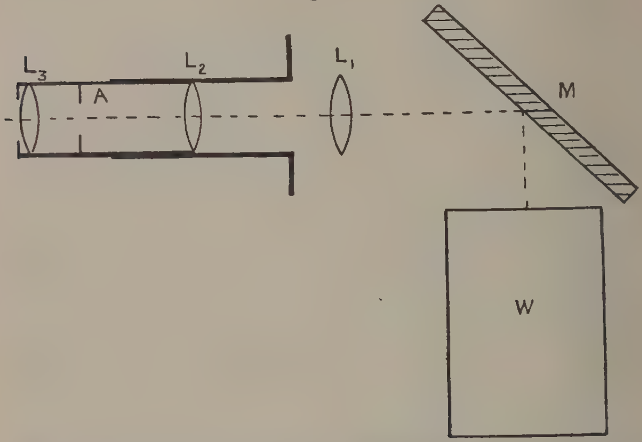
* 'Electrician,' Sept. 14th, 1906, for some notes by the author on this point.

This is accomplished as follows.

Fig. 4.



The image of the wedge, as formed by the mirror M, is outside the focal length of the convex lens $L_1$, and this together with the lens $L_2$ forms a real image of the illuminated surfaces in the plane of the aperture at A. The eye of the observer applied to the third lens $L_3$ sees a magnified image of the aperture and the illuminated surfaces—both simultaneously in focus.

If, now, an observer is comparing, say, a red source with a green one, the field of view appears to him as shown in fig. 5 *a*, when the lens $L_1$ is stationary. He then adopts as photometric balance the position of the photometer-wedge in which the red and green fields appear equally bright.

The element of flicker may, however, be introduced by a modification of the method due to Rood *. The lens, $L_1$, is mounted on springs and attached to a cord passing over a pin mounted eccentrically on the pulley of a small electric motor. When the motor is caused to rotate the lens oscillates to and fro and the boundary between the photometrical surfaces appears to oscillate with it. A band of flicker is thus produced, and the field of view assumes the appearance shown in fig. 5.

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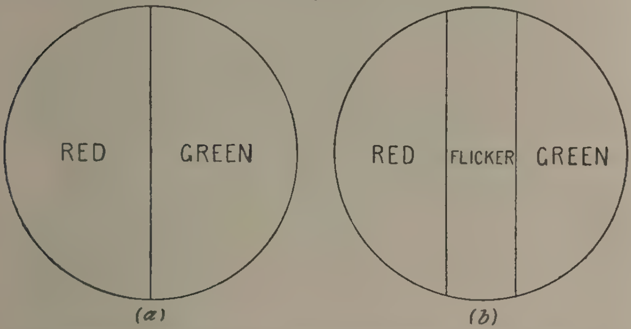
* Amer. Journal of Science, 1899, p. 194.

The observer may then judge the position of photometric balance by observing the cessation of flicker in the band. Or he may drive the motor so fast that all flicker disappears and the intermediate band merely assumes a colour intermediate between that present in the two adjacent portions of the field of view. He may then use this intermediate colour to assist him in his decision as to the exact point when all three sections of the field of view appear equally bright.

Fig. 5.



The question may now be raised whether, when using this instrument, the results of using the Equality of Brightness and Flicker methods is the same.

The author has found that for widely different colours and for certain portions of the retina this is not the case *. Under the conditions present in this instrument, however, the agreement was very close unless the colour difference was very marked indeed. This is illustrated by the results shown in the following:—

| Nature of Lights compared. | Results by Flicker method. | Result by Equality of Brightness method. |
|---|---|---|
| White to White ......... | 1·031 | 1·034 |
| White to Apple-Green. (impure) | 0·911 | 0·906 |
| White to deep Green ... | 1·21 | 1·08 |
| White to deep Red ...... | 2·23 | 2·09 |

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* Proc. Phys. Soc. vol. xx.

When comparing lights of the same colour the sensitiveness of the photometer seemed to be very much the same for both methods, but when the comparison of sources of light differing widely in colour was attempted, the Flicker method gave rise to the most consistent results.

It seems possible that the sensitiveness of a photometer of this type might be improved by introducing the element of *contrast* in the manner shown in fig. 6 (Pl. III.)

Each of the photometrical surfaces is shown divided into 3 equal and similar portions. The portions 2 and 5 are intended to be as white as possible. The portions 1 and 6 are tinted with a neutral wash of such depth as to reduce the coefficient of reflexion from the surface by say 4 per cent., while 3 and 4 are intended to be intermediate in shade between 1 and 5 and 2 and 6 respectively.

Now if the Equality of Brightness method is used, the observer attempts to set the wedge so that 3 and 4 appear equally bright. The contrast in light and shade exhibited between 1 and 2 and 6 and 5 respectively should then be the same.

If, now, the wedge is rotated slightly in a clockwise direction, the contrast between 1 and 2 becomes *more* marked but that between 6 and 5 *less* marked than before.

We have thus a *double* change such as occurs in the Lummer-Brodhun contrast photometer and a corresponding gain in sensitiveness. If the Flicker method is used a corresponding effect will exist.

The author has endeavoured to facilitate the comparison of heterochromatic sources of light by applying Crova's method.

As the temperature of an incandescent solid is raised, the luminous radiation in the form of waves of short wave-length increases more rapidly than the luminous radiation of great wave-length.

According to the principle discovered by Crova*, however, the luminosity in the spectrum of such a source in the neighbourhood of $\lambda = 0{\cdot}582\mu$ is a measure of the total luminosity of the source. Crova therefore proposed to view the surfaces illuminated by two heterochromatic sources through a solution allowing only rays in this portion of the spectrum to pass. This solution was composed of :—

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* *Comptes Rendus*, vol. cxix. no. 16, p. 627.

Anhydrous ferric chloride, 22·32 gms.
Crystallized nickel chloride, 27·19 gms.
Distilled water, 100 cc.

It must be noted, however, that the principle on which Crova's method depends, only applies rigidly to an incandescent body yielding a continuous spectrum.

In the case of a luminescent vapour yielding a spectrum in which the luminosity is concentrated in isolated bright lines, the method is inapplicable.

Yet, since the portion of the spectrum which Crova's method proposes to utilize is by far the most valuable portion from an illuminating point of view (*i. e.* the portion which, for a given expenditure of energy, produces the greatest sensation of brightness), the method may prove satisfactory even in many cases of this character.

The chief difficulty which the author has found in applying this method lies in the fact that, if a sufficient depth of solution is employed to restrict the light passing to a very narrow region of the spectrum, and hence to render the colours of the two illuminated surfaces identical, the luminosity of the surfaces becomes, as a rule, inconveniently low. The only case, in fact, in which the author succeeded in obtaining an apparently perfect colour match without undue loss of illumination, was when comparing an arc lamp against an incandescent mantle.

When a *flicker* was utilized, however, it was found that a depth of 3 millimetres of solution, which, while most transparent to yellow rays, actually allowed rays from $\lambda=0{\cdot}52$ to $\lambda=0{\cdot}68\mu$ to pass, rendered the comparison of most sources distinctly easier.

The following are a few cases in which this method was applied.

Carbon Glow-lamp compared with Nernst lamp.
Argand Gas-burner „ „ Incandescent Mantle.
White Arc-light „ „ Incandescent Mantle.
White Arc-light „ „ Argand Gas-burner.
Flame Arc-lamp (Excello carbons) compared with Argand Gas-burner.
Flame Arc-lamp (Bremer carbons) „ „ Argand Gas-burner.

In all these cases the means of a set of photometrical results obtained with and without the Crova screen agreed very closely, in the first two cases within 2 per cent.

The unsteady nature of the arc lights rendered very exact comparisons with these lights impossible, but in this case also the difference seemed to be, at any rate, less than 5 per cent.

All these sources of light, however, yield a continuous spectrum. It is true that some isolated bright lines, superimposed over the main spectrum of the arc, serve to increase the efficiency of the flame carbons. Yet the brightest of these lines were congregated in the yellow and orange and would, therefore, not seriously affect the applicability of Crova's method.

Quite otherwise were the results obtained when comparing a mercury-vapour lamp against a carbon-filament glow-lamp. The luminosity of this lamp was concentrated in three isolated bright lines in the yellow, green, and violet respectively, practically no red rays being present.

It was found that the depth of solution utilized in the experiments described above not only failed to very noticeably improve the colour difference between the sources but also seriously affected the photometric results obtained.

The use of yellow screens in order to facilitate the photometric comparison of heterochromatic sources of light is certainly only justifiable in the case of illuminants yielding continuous spectra.

Even in this case it is essential that the yellow screen should exhibit absorbing properties similar to those of Crova's solution, namely minimum absorption in the neighbourhood of $\lambda = 0{\cdot}58\mu$, and a uniformly increasing absorption on either side of this point.

## Discussion.

Mr. Patterson complimented the author upon his paper and experiments. With reference to the cosine instrument he asked if the cosine law was trustworthy at very acute incidence of the light. The unequal sensitiveness of different parts of the retina accounted for the fact that it was possible to obtain different balance-points by fixing attention on the different sides of a Ritchie wedge.

Mr. J. S. Dow, in reply, said he had only worked with photometers with wedges of 90°, but he was certain that the cosine law would not hold at acute incidence on account of the large amount of regular reflection.

## IV. *On Magnetic Oscillators as Radiators in Wireless Telegraphy.* *By* J. A. Fleming, *D.Sc., F.R.S.*[*]

An oscillator of the open or Hertzian type is commonly called an *electric oscillator* because the effects produced in the external field are to a large extent determined by the potential of the free electric charges which alternately make their appearance on the open ends. If, however, the oscillator consists of a metallic circuit completed by a condenser, the plates of which are very near together, the effects in the external space are mostly or entirely determined by the current in the circuit and little if at all by the condenser-plate charges, because these being of opposite sign end near together neutralize each other's effects in the field. Such a closed or nearly closed circuit is called a *magnetic oscillator*. In the case of an open oscillator or simple doublet Hertz showed that if $\phi$ is the maximum electric moment, or product of the maximum static charge on each end and the distance between them, and if $\lambda$ is the wave-length of the emitted radiation, the energy radiated by the oscillations in ergs per period is given by the expression

$$E=\frac{16\pi^4\phi^2}{3\lambda^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This expression can be reduced to a more convenient form for practical measurement as follows :—

Let C be the capacity in microfarads of each sphere or half of the oscillator with respect to the other, and V be their maximum potential difference in volts before discharge, then $9\times10^5\,C\,V/300=3000\,C\,V$ is the maximum value of the charge in electrostatic units.

Hence if $l$ is the length of the oscillator in cms., we have for the electric moment,

$$\phi=3000\,C.V.l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the current at the centre of the oscillator have a maximum value A reckoned in amperes, and let $a$ be the root-mean-square or effective value in amperes as measured by a hot-wire ammeter.

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[*] Read October 25, 1907.

Then, if the oscillations are sinoidal in form, undamped or persistent, and of frequency N, we have

$$A=2\pi\,N\,C\,V/10^6, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

$$a=\frac{A}{\sqrt{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Hence the energy E radiated per period in ergs is given by

$$E=4\pi^2\,10^8\,\frac{l^2}{\lambda^2}\,\frac{A^2}{N}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

and the power W in watts by

$$W=40\pi^2\frac{l^2}{\lambda^2}A^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Remembering that $\pi^2=9\cdot87$ and $N\lambda=3\times10^{10}$, we can write the above formulæ in the convenient form

$$E=0\cdot2632\frac{l^2}{\lambda}a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

$$W=789\cdot6\,\frac{l^2}{\lambda^2}a^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

The value of the ratio $l/\lambda$ can be determined experimentally for any oscillator. For a simple linear oscillation it approximates in value to 0·4. Hence for such an oscillator radiating undamped waves, the power radiated in watts is given by the expression $W=126a^2$. Thus, for instance, if the effective value of the current at the centre of the oscillator is 2 amperes, the radiation will be half a kilowatt. If we then consider the case of a perfectly closed oscillation circuit having an area S and traversed by undamped or persistent oscillations which have a maximum value A and R.M.S. value $a$ reckoned in amperes, it can be shown that such a closed circuit also radiates energy, and we can derive a similar formula for the radiation to that given by Hertz for the open circuit. The author has shown that if M is the maximum magnetic moment of such a closed circuit *,

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* See 'The Electrician,' vol. lix. p. 936 (1907). A series of articles on "The Elementary Theory of Electric Oscillators," by J. A. Fleming.

viz., the product of its area S and the maximum value of the alternating current (reckoned in electromagnet measure) in it; and if $\lambda$ is the wave-length of the radiation emitted by it, and E the radiation in ergs per period, then for the magnetic oscillator we have

$$E=\frac{16\pi^4 M^2}{3\lambda^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

But by definition $M=AS/10$, and if the current is sinoidal $a=A/\sqrt{2}$, hence we have

$$E=10{\cdot}4\frac{S^2a^2}{\lambda^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Furthermore, if the oscillations are persistent the power in watts W radiated is given by

$$W=31{,}200\frac{S^2a^2}{\lambda^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The formulæ (8) and (11) can be written in the form

$$\left.\begin{array}{ll} W=87\times10^{-20}\,l^2a^2N^2 & \text{(open or electric oscillator),} \\ W=\ 4\times10^{-38}\,S^2a^2N^4 & \text{(closed or magnetic oscillator).} \end{array}\right\} \quad (12)$$

These last two formulæ show us that in the case of the open or electric oscillator the power radiated varies as the square of the current in it, and as the square of the frequency, whereas in the case of the closed or magnetic oscillator it varies as the square of the current but as the fourth power of the frequency. Accordingly, in the case of the magnetic oscillator the radiation is small unless the frequency is very high.

For the purposes of theory, it is most convenient to consider a closed oscillatory circuit as made up of a series of Hertzian or dumbbell oscillators placed in series with their electric poles of opposite sign overlapping. We can then easily determine the nature of the electric and magnetic field produced by it by the summation of those due to the elementary oscillators. As the writer has given lately in another place a discussion of the problem and furnished the expressions for the electric and magnetic forces at any point in the field of such a closed oscillator and the derivation of the formula (10) for the radiation, it is unnecessary to go over the ground again here. (See 'The Electrician,' vol. 59, p. 936 *et seq.*)

The object of the following experiments was to obtain some information as to the actual distance-effects of closed oscillators of certain dimensions, and to ascertain whether true radiation was at all influential in producing the inductive action of one closed oscillatory circuit on another at moderate distances, or if the whole of the action could be accounted for by ordinary electromagnetic or Faradaic induction.

The following circuits were employed for this purpose :—

Two pair of square circuits were constructed by winding a few turns of highly insulated copper wire round a wooden cross.

One pair had sides 8 feet in length, each frame carrying two coils of 5 turns of 7/21 stranded insulated copper wire. These will be referred to as the "large square coils."

One pair had sides 2 feet in length, each frame carrying one coil of 8 turns of the same sized wire. These are referred to as the "small square coils."

The experiments consisted in setting up continuous or undamped oscillations in one of these coils by means of a Poulsen arc and then measuring the current created in the corresponding coil placed at a distance by means of the Author's oscillation-valve or glow-lamp detector. The two coils were placed at various distances and in various respective positions. The coils themselves were erected or placed in the large quadrangle of University College, London. As the arc apparatus had to be placed indoors in the Electrical Laboratory it was necessary to connect the coil used as a transmitter or primary coil with the oscillation-producing apparatus by means of a long pair of insulated parallel copper (7/21) wires 14 cms. apart and $27\frac{1}{2}$ feet or 836 cms. long.

The first step was to measure the inductance of these wires for high frequency currents. Anderson's bridge method was employed with a commutator as described by the Author *.

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* See J. A. Fleming, "A Note on the Measurement of Small Inductances and Capacities," Phil. Mag. May 1904, p. 586.

See also J. A. Fleming and W. C. Clinton, "On the Measurement of Small Capacities and Inductances," Phil. Mag. May 1903, p. 493.

The inductance of the parallel wires or tails above mentioned was found to be 50,910 cms., and that of one large coil of 5 turns on the 8-foot square together with the tail (L) was 310,740 cms. Hence the inductance of the coil itself of 5 turns was 259,830 cms. The inductance of the two coils each of 5 turns in parallel was 198,350 cms., and in series 779,300 cms.

The inductance of each of the small or 2-foot square coils was in the same manner found to be 116,200 cms.

In series with these coils was placed a condenser having a capacity (C) of 0·0026 mfd. and the coil and condenser shunted across a 400-volt Poulsen arc.

The arc current was generally about 8 amperes, the potential difference of the arc terminals ($V_0$) 260 volts (continuous). The potential difference ($V_1$) (R.M.S. value) of the condenser terminals was 1580 volts as measured by an electrostatic-voltmeter and the R.M.S. current (*a*) in the square coil was 4·22 amperes as measured by a hot-wire ammeter. The frequency N of the oscillations is given by the formula $N=(5{\cdot}033\times10^6)/\sqrt{CL}$. Hence if $C=0{\cdot}0026$ mfd. and $L=310740$ cms., we have $N=177{,}100$. If the R.M.S. current (*a*) in the coils were strictly sinoidal in waveform, the value should be calculable from the formula,

$$a=\frac{2\pi}{10^6}CVN$$

where $V=\sqrt{V_1^2-V_0^2}=\sqrt{(1580)^2-(260)^2}=1560$ volts. Hence we should have

$$a=\frac{44}{7}\times\frac{26}{10^{10}}\times1560\times177100=4{\cdot}52 \text{ amperes.}$$

The actual measured value was 4·22 or 7·3 per cent. less, which may be accounted for by the known fact that the actual wave-form of the current in the coil is not by any means truly sinoidal.

The wave-length λ corresponding to a frequency of 177100 is nearly 169,000 cms., or rather more than 1 mile in length.

The total area S of the large coils using the 5-turn circuit is $5\times(243{\cdot}2)^2$ cms.$=295{,}731$ sq. cms., and the value of $S^2a^2$ is therefore $(295731)^2 \times (4{\cdot}22)^2 = 156 \times 10^{10}$. Also $(177100)^3 = 554523 \times 10^{10}$. For a closed square radiator the energy in ergs radiated per period is given as already mentioned by the expression

$$E = 10{\cdot}4 \frac{S^2a^2}{\lambda^3}.$$

Hence we have in this case for the power W radiated in ergs per second,

$$W = \frac{10{\cdot}4 \times 156 \times 177000}{554523} = 522 \text{ (nearly)}$$

or about 52 microwatts.

This is the energy sent out per second through the surface of a sphere described round the oscillator the radius of which is large compared with the wave-length of the oscillator in question.

It is evident therefore that the true radiative power of a closed circuit of the above dimensions is extremely small compared with that of an open oscillator of the same order of linear dimensions.

Hence when a closed circuit traversed by a high frequency current acts to produce secondary currents in another similar circuit at less than a wave-length distance from it, the chief part of the effect cannot be due to true radiation or detached energy but is the result of ordinary electromagnetic induction, or the creation of secondary currents produced by the ebb and flow or pulsation of the lines of magnetic force permanently connected with the primary circuit*.

Nevertheless experiment shows that such circuits may act appreciably upon one another at very considerable distances, when the primary circuit of one is traversed by a high frequency current, and the circuits are in resonance with each other.

The old form of wireless telegraphy based upon such mutual induction between closed coils, as in the arrangement suggested by Trowbridge and Stevenson, or the parallel wire method of Preece, Rathenau and others, employed relatively low frequency currents because the receiver used was the telephone, and the frequency was therefore limited by the range of audition. Even when the circuits were put in resonance, as by Lodge, the difficulties connected with the disturbance of commercial telephony restricted the possibilities of its use apart from other considerations.

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* Moreover, just as in the case of the Hertzian oscillator true radiation or detachment of energy only takes place about $\frac{1}{4}$ or $\frac{1}{2}$ a wave-length from the oscillator, so in the case of the closed or magnetic oscillator true radiation effects can only be produced at a similar distance from it.

One object of the present experiments was to obtain data for an opinion whether such inductive wireless telegraphy could be improved by the use of electric oscillations of high-frequency in completely non-earthed circuits so that no disturbance of telephones or telegraphs could possibly arise combined with the use of a suitable oscillation detector. With this end in view, a closed primary-circuit had created in it continuous or undamped electric oscillations, and its inductive and radiative effect upon a similar closed receiving circuit was measured by means of the Oscillation Valve or Glow-Lamp Detector first described by the Author in 1904.

It consists of a 12-volt carbon filament glow-lamp with a metal cylinder surrounding the filament sealed into the bulb. The cylinder is carried on a platinum wire sealed through the glass. The filament is incandesced by a 6-cell secondary battery. If the terminals of a condenser inserted in an oscillatory circuit are connected, one to the cylinder and one to the negative terminal of the lamp filament, and if an ordinary movable coil galvanometer is inserted in this circuit, then when oscillations occur in the condenser the glow-lamp acts as an electrical valve and the galvanometer is traversed by a continuous current which bears a definite relation to the potential difference of the condenser terminals and therefore to the current into it*.

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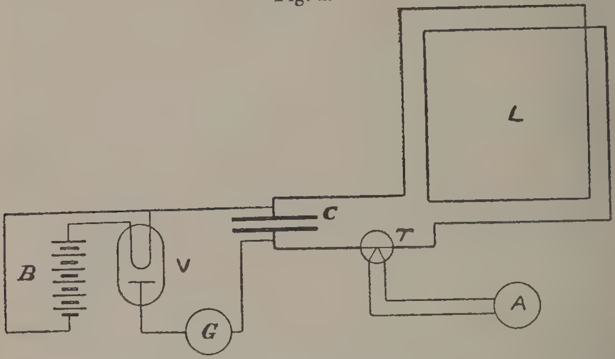
* This glow-lamp detector or oscillation valve was first described by me in a British Patent Specification, No. 24,850 of 1904, and in a Paper to the Royal Society (see Proc. Roy. Soc., vol. lxxiv. p. 476) in February 1905. Also in a United States Patent Specification, No. 803,684 of 1905, and in a Paper to the Physical Society of London (see Phil. Mag. May 1906), as well as in a book on the Principles of Electric Wave Telegraphy, pubished in May 1906. These publications apparently failed to make it known in the United States, for in October 1906 Dr. Lee de Forest described to the American Institute of Electrical

To calibrate the valve the following arrangement was employed. An oscillatory circuit was arranged consisting of one of the small square coils L (see fig. 1) and a condenser or Leyden jar C in series with it. In this circuit at one point was inserted a fine constantan wire having a thermo-junction (T) of iron and bismuth or bismuth and tellurium in contact with it. The arrangement actually employed was either the thermo-junction used with the Author's Cymometers or else the thermoelectric receiver described some time ago* (see fig. 1).

Fig. 1.



The galvanometer A associated with the junction was a 5-ohm single-pivot Paul galvanometer. The arrangement was calibrated by first passing measured continuous currents through the constantan wire and observing the corresponding scale-readings of the Paul galvanometer. The curve plotted out proved to be very approximately a parabola, the galvanometer deflexion being very nearly proportional

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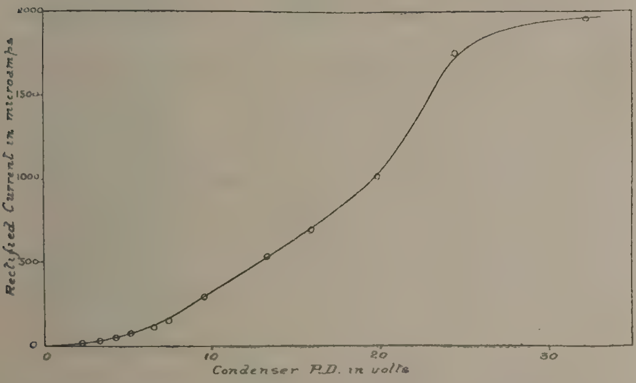
Engineers precisely the same device, viz., a carbon filament glow-lamp having a plate sealed into the bulb as a detector for electric oscillations, as his own invention and re-christened this two-year old appliance an *audion*. No re-christening, however, affects the fact that the appliance had been in use by me long previously and especially described as a receiver for wireless telegraphy.

[*] See "Radiation from Bent Antennæ," by J. A. Fleming, Phil. Mag. December 1906, p. 592.

to the square of the heating current sent through the constantan wire. When electric oscillations were set up in the circuit they heated the constantan wire and caused a deflexion of the Paul galvanometer from which by the previous calibration their mean-squre value could be determined. In addition the oscillation valve V joined in as shown in fig. 1 passed a continuous current through the galvanometer G in its circuit, and readings were taken of this current in microamperes corresponding to various measured values of the oscillatory current in the square circuit which passed through the constantan wire.

The object of this calibration was to be able to use the valve subsequently as a means of measuring the oscillatory

Fig. 2.

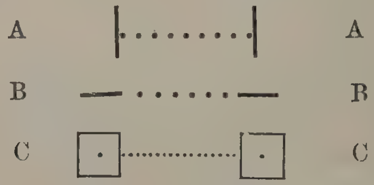


current induced in the circuit when it had a value less than that which could be easily measured by the thermojunction, for when associated with a suitable mirror-galvanometer it provides the means of measuring a high frequency current of a few microamperes. Moreover it does not introduce any extra resistance into the circuit, as would have been the case if the constantan wire and themojunction had been permanently inserted into the circuit. Observations showed that the galvanometer current did not increase proportionately to the condenser plate potential difference, but tended to a "saturation" value as shown by the curve in fig. 2.

The large (8-feet) square coils were then set up in the quadrangle of University College, one being used as a transmitter and the other as a receiver circuit.

The transmitter coil, in series with a suitable condenser, was connected by the long parallel wires or tails with a Poulsen arc in the adjacent electrical laboratory, and persistent oscillations excited in it. The similar receiver circuit at a distance had a condenser of equal capacity in series with it and a variable inductance for tuning. The oscillation valve had its terminals connected to the receiver circuit condenser-plates and wires taken from it to a mirror galvanometer placed in a distant room. The transmitter coil had then persistent oscillations set up in it by the arc, and the transmitter and receiver coils were placed at various distances and in various relative positions, and the current created in the received circuit was in each case measured by the galvanometer and valve.

When either of the square coils were placed with their planes normal to the line joining their centres and vertical to the earth surface, we shall call this position A. When either of them were placed with plane parallel to the line joining their centres and vertical to the earth we shall call the position B, and when they were placed with plane parallel to the surface of the earth we shall call this position C. Thus if we represent the coils as seen in plan or looking down on them from above we can denote these positions as follows:—



The dotted lines represent the line joining their centres. The following experiments were then made :—

Exp. I. *Effect of varying the position of the coils.*—The two 8-feet coils of 5 turns each were set up with centres 60 feet apart. The transmitter current was 5·4 amperes. The capacity in the transmitter circuit 0·0026 mfd. The receiver was tuned to a frequency of 164,000 in syntony with the transmitter and undamped oscillations were set up in the latter by the arc. The electric arc was adjusted to keep a constant high-frequency current in the transmitter circuit of 5·4 amperes. The arc current itself was 8 amperes and arc P.D. from 260 to 320 volts.

The transmitter was put successively in positions A, B, and C, and in each case associated with receiver in positions A, B, and C, and the receiver or secondary current in milliamperes being measured. The results are tabulated in Table I.

Table I.

| Transmitter Position. | Receiver Position. | Secondary Current in milliamperes. |
|---|---|---|
| A | A | 24·16 |
| " | B | 10·77 |
| " | C | 20·98 |
| B | A | 5·25 |
| " | B | 20·45 |
| " | C | 18·10 |
| C | A | 6·56 |
| " | B | 22·58 |
| " | C | 37·00 |

When the coils were in position C their planes were 6 inches above the ground.

The figures in Table I. show that when both coils were in position C the receiver current exceeded that in any other position, and that although the receiver current was small with the coils in the positions of zero mutual induction, viz. (A.B.), (B.A.), or (C.A.), it was not small in the position (A.C.).

In the case of two small coils placed at a distance large compared with their dimensions the magnetic induction for a steady field should by theory be twice as great in the position (A.A.) as in the position (B.B.), and should be the same in position (C.C.) as in position (B.B). Also zero in the 6 positions (A.B.), (B.A.), (C.B.), (B.C.), (A.C.), (C.A.).

But when using the high-frequency primary current the secondary current is still large in the cases (A.C.),( B.C.), and (C.B.), viz., when either the receiver or transmitter coil has its plane parallel to the earth's surface.

Subsequent results showed that the height of the coils above the ground affected this result considerably. Nevertheless the exalted effect in position (C.C.) is interesting and convenient because it is easy to construct large circuits out of doors in positions (C.C.) and very much less easy without the aid of tall masts to make them in the positions (A.A.) or (B.B.).

Exp. II. *Effect of varying distance. Coils vertical.*—The same pair of coils were set up on positions (A.A.) and their distances varied. The receiver circuit was tuned to a frequency of 162,000, and the primary current was 4·7 and 5·4 amperes in two sets of observations respectively.

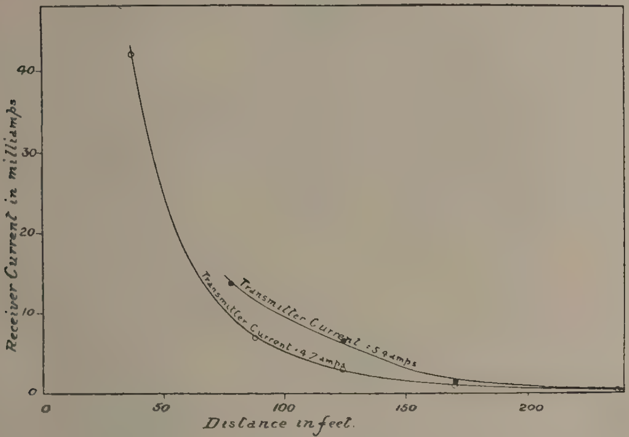
Table II.—Coils in position (A.A.).

| Distance of Centres in feet. | Primary Current in amperes. | Secondary Current in milliamperes. |
|---|---|---|
| 78·0 | 5·4 | 13·7 |
| 123·5 | „ | 6·46 |
| 170·5 | „ | 1·17 |
| 237·0 | „ | 0·12 |
| 36·2 | 4·7 | 41·9 |
| 88·0 | „ | 6·99 |
| 123·2 | „ | 3·05 |
| 169·7 | „ | 1·01 |
| 237·0 | „ | 0·11 |

The results are plotted in the curve in fig. 3.

The curve shows that the inductive effect decreases at first very rapidly with the distance, less rapidly than the inverse cube but rather more than the inverse square of the distance.

Fig. 3.



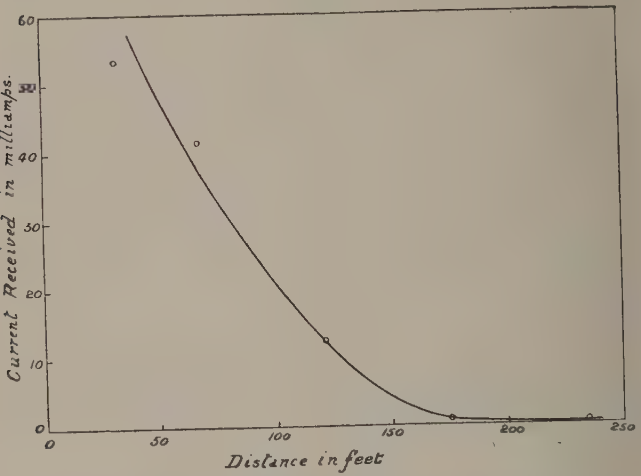
Exp. III. *Effect of varying distance. Coils horizontal.*—The same coils were set in positions (C.C.) with centres 6 inches above the ground. The receiver circuit was tuned to a frequency of 162,000, and the primary current was 5·4 amperes.

Table III.—Coils in position (C.C.).

| Distance of Centres in feet. | Primary Current in amperes. | Secondary Current in milliamperes. |
|---|---|---|
| 33 | 5·4 | 53·1 |
| 68 | " | 41·4 |
| 121 | " | 12·4 |
| 175 | " | 0·57 |
| 236 | " | 0·28 |

Here again the current falls off at a rate between the inverse cube and inverse square, but the secondary current has a much greater absolute value for the same distance than with the coils in positions (A.A.). The results are set out in the curve in fig. 4.

Fig. 4.



Exp. IV. *Effect of varying distance. Small coils horizontal and 6 inches above the ground.*—A similar set of measurements were made with the small coils 2 feet in side and having 8 turns on each coil. The primary current was 5·4 amperes. The receiver circuit was tuned to a frequency of 214,400 in syntony with the transmitter current.

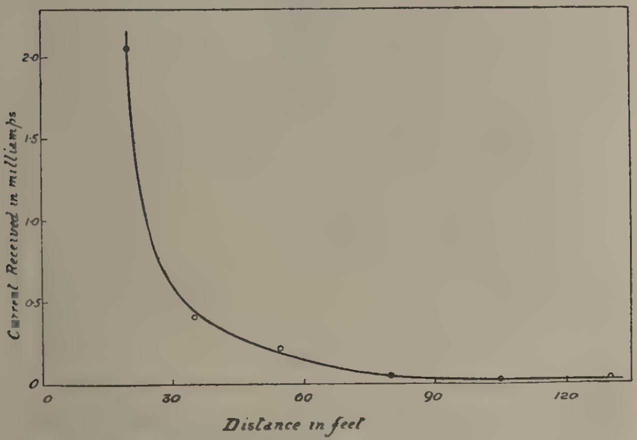
The coils were placed at various distances and the receiver currents measured.

Table IV.

| Distance of Centres in feet. | Primary Current in amperes. | Secondary Current in milliamperes. |
|---|---|---|
| 20 | 5·4 | 2·045 |
| 35 | ,, | 0·402 |
| 55 | ,, | 0·223 |
| 80 | ,, | 0·045 |
| 105 | ,, | 0·027 |
| 130 | ,, | 0·034 |

The same law of variation holds good in this case also as shown by the curve in fig. 5.

Fig. 5.



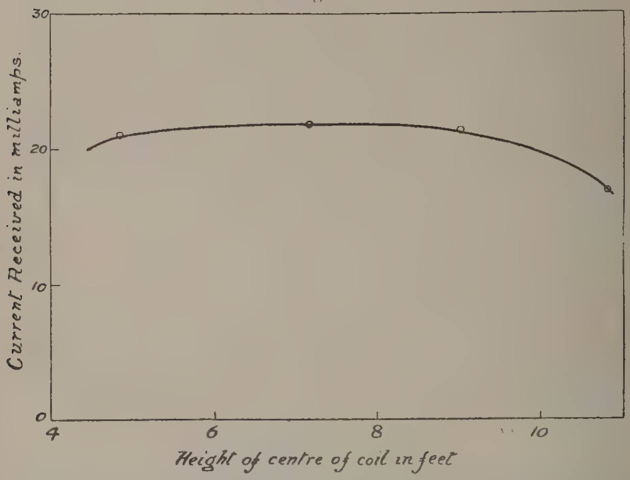
Exp. V. *Effect of varying height above the ground. Coils in position* (A.A.) *vertical.*—The large coils were employed and raised step by step to various heights above the ground. The primary current was 5·4 amperes. The receiver was tuned to a frequency of 162,000. The distance of the coils was 68 feet. The coils had their planes vertical and parallel to each other except in the last case when they were placed horizontal or in positions (C.C.) for comparison.

Table V.

| Height of centre of Coil above ground, in feet and inches. | | Receiver Current in milliamperes. | Coil Position. |
|---|---|---|---|
| ft. | ins. | | |
| 4 | 10 | 20·98 | Coils vertical (A.A) |
| 7 | 2 | 21·76 | „ „ „ |
| 9 | 0 | 21·50 | „ „ „ |
| 10 | 10 | 16·84 | „ „ „ |
| 0 | 6 | 31·08 | Coils horizontal (C.C.) |

Here again is shown the improved effect of placing the coils with their planes parallel to the earth's surface. These observations are plotted in the curve shown in fig. 6.

Fig. 6.



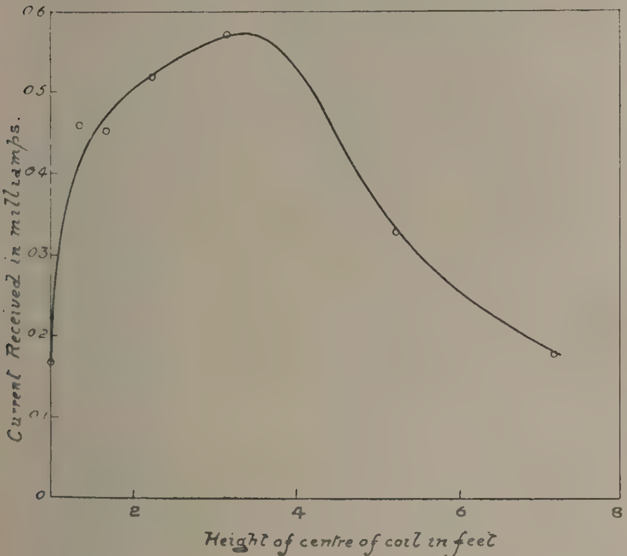
Exp. VI. The same measurement was made with the small coils 2 feet square of 8 turns, placed at a distance of 60 feet. The primary current was kept at 5·4 amperes. The receiver was tuned to a frequency of 214,000 and the secondary current was measured for various heights of the coils above the ground, the coils being vertical and in positions (A.A.).

Table VI.

| Height of centre of Coil above the ground, in feet and inches. | | Secondary Current in milliamperes. |
|---|---|---|
| ft. | ins. | |
| 1 | 0 | 0·166 |
| 1 | 4 | 0·460 |
| 1 | 8 | 0·452 |
| 2 | 3 | 0·515 |
| 3 | 2 | 0·570 |
| 5 | 3 | 0·327 |
| 7 | 2 | 0·176 |

The results of the last two experiments is to show that for constant primary current the receiver current is a maximum for a certain height of the coils above the earth's surface. The observations are delineated in fig. 7.

Fig. 7.



It is obvious, however, that all the measurements in which the coils are placed horizontal and at varying heights above the ground need caution in interpreting them. When the coil is placed with its plane parallel to the earth, currents must be induced in the soil, and the effective inductance of the coil and therefore the frequency of the current must be altered. The nearer the ground the less is the effective inductance of the coil, but on the other hand the capacity is increased. Hence, even if the primary high-pressure current is kept constant, there will be variations of frequency which affect the inductive action of the primary on the secondary. That such is the case was shown by the results of a further set of measurements with the large coils, the power taken up in the arc and its power-factor being observed at the same time.

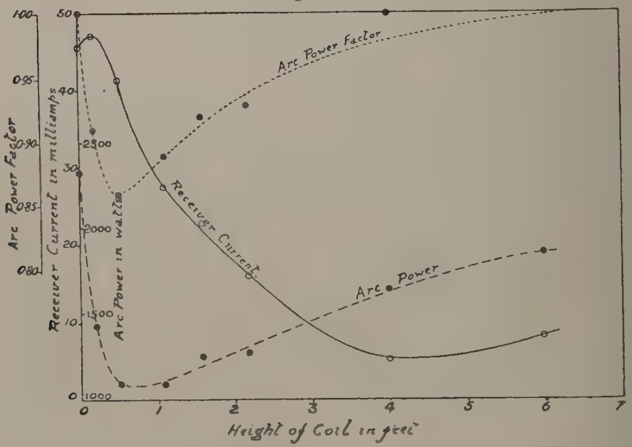
Exp. VII.—Large square coils placed parallel to the earth. Distance 68 feet. Receiver tuned to a frequency of 162,000 approximately in syntony with the transmitter. The coils were raised to various heights above the ground and the secondory current measured. The transmitter current was kept constant at 5·4 amperes.

Table VII.

| Height of Coils above the ground, in feet and inches.<br>ft. ins | Power taken up in Arc in watts. | Power factor of Arc. | Secondary Current in milliamperes. |
|---|---|---|---|
| On ground | 2328 | 1·0 | 45·8 |
| 0 2 | 1424 | 0·91 | 47·4 |
| 0 6 | 1096 | 0·86 | 41·2 |
| 1 1 | 1104 | 0·89 | 27·7 |
| 1 7 | 1248 | 0·92 | 22·5 |
| 2 2 | 1272 | 0·93 | 16·0 |
| 4 0 | 1640 | 1·00 | 5·18 |
| 6 0 | 1856 | 0·98 | 8·03 |

The above observations are reduced to graphic form in fig. 8.

Fig. 8.



They show how much the receiver current varies with the height of the coils above the ground, the maximum value being reached when the coils are about 2 inches above the surface. Also they show that when the coils are laid flat on the ground there is a great increase in the power taken up by the arc.

Raising the coils from 6 inches to 6 feet above the ground decreased the secondary currents to one-fifth.

Exp. VIII. *Comparison of coils of different areas.*—The two large or 8-feet coils were compared with the two small or 2-feet coils. In both cases the distance between primary and secondary was 79 feet, and the coils were parallel to the earth and at 6 inches above the surface. In both cases the primary current was 5·4 amperes.

The small coils had 8 turns and were tuned to a frequency of 214,400.

The large coils had 5 turns and were tuned to a frequency of 162,000.

Table VIII.

| Coils used. | Number of turns. | Frequency. | Secondary Current in milliamperes. |
|---|---|---|---|
| Small.<br>2 feet in side ... | 8 | 214,400 | 0·082 |
| Large.<br>8 feet in side ... | 5 | 162,000 | 27·4 |

The large coil therefore produces 340 times the effect of the small coil.

Exp. IX. *Comparison of the same coils with different numbers of turns.*—The two large coils were wound with two circuits each of 5 turns and they could be joined in series so as to make a coil of 10 turns. The coils were placed parallel to the earth and at a distance of 6 inches above it. The primary current used in both cases was 5·4 amperes. The inductance of the primary circuit was 380,000 cms., and when 10 turns were used it was 1,000,000 cms. The power taken up in the arc therefore was not the same in the two cases.

TABLE IX.

| Number of turns on Coil. | Power taken up in Arc in watts. | Frequency of Oscillations. | Secondary Current in milliamperes. |
|---|---|---|---|
| 5 | 2495 | 162,000 | 27·4 |
| 10 | 1537 | 99,600 | 21·4 |

The increase in the number of turns by increasing the inductance lowers the frequency of the oscillations, and this lowers the power taken up in the arc, and therefore the secondary current is reduced roughly in proportion to the decrease in frequency. In all the above experiments the Poulsen arc was used as a generator of undamped oscillations. It was considered desirable, however, to compare the effects obtained in this case with those when using a spark method of exciting intermittent oscillations.

The large square coils were placed vertical and facing each other in positions (A.A.) and at 68 feet apart. They were tuned to a frequency of 162,000. Oscillations were then excited by an arc and a spark and the primary and secondary currents measured and volt amperes given to the generator in each case, as follows:—

TABLE X.

| Generator. | Volts. | Amperes. | Volt amperes. | Primary Current in amperes. | Secondary Current in milliamperes. |
|---|---|---|---|---|---|
| Spark ......... | 12·2 | 2·6 | 31·7 | 2·15 | 0·56 |
| Arc ......... | 260 | 8·0 | 2080 | 5·4 | 21·0 |

Roughly speaking, therefore, 70 times more power was being expended on the arc than on the induction-coil used to create the spark, but the secondary current only increased some 38 times.

This seems to indicate that the intermittent spark method of exciting the oscillations is not inferior but rather superior in efficiency to the arc method, for the particular purpose here considered.

The conclusions to be drawn from this preliminary set of experiments are that the inductive effects observed between the closed oscillatory circuits used are in the main due to electromagnetic induction and not to true radiation. Secondly, that the most advantageous position of the coils is with the plane of the coils parallel to the earth's surface and at a small distance above it. Thirdly, that the increase in the area of the coils is especially advantageous, and that therefore the coils used should consist of single turns of wire enclosing as large an area as possible. Fourthly, that the inductance and capacity of the circuit must be kept low to keep up the frequency. Lastly, the spark method of exciting the oscillations can be advantageously employed.

The best method, therefore, for constructing and using such closed circuit antennæ for high-frequency closed circuit wireless telegraphy which seems to be indicated by the foregoing experiments is as follows :—

In some large field set up a number of short wooden posts having telegraphic insulators attached to them. Round these lay a number of wires, which may be bare copper wires, so as to form a closed circuit of one turn of a number of parallel wires, the circuit being say a foot or two from the ground, or if cows, horses, or sheep are kept in the field it may be better to keep it up 7 or 8 feet from the ground. Complete this circuit by a condenser and spark-gap and make provision by a high-tension transformer for exciting oscillations in the circuit.

In a similar distant and syntonised circuit include any of the receiving devices now employed in radio-telegraphy, and signals can be transmitted and received without high antennæ and without interruption to commercial telephonic circuits. Such closed horizontal receiving circuits will not pick up signals from the vertical antenna transmitters commonly used.

The method suggested is in fact the old form of magnetic induction space-telegraphy, but conducted with electric oscillations in place of the low-frequency currents obtained by an ordinary alternator or interrupted battery-current. On the other hand, direct experiment with low-frequency alternating currents of 5·4 amperes passed through the small coils used in these experiments, showed that no inductive effects detectable at more than a few yards at most were apparent when using an ordinary telephone as a receiver. Even if the whole of the inductive effect is due to simple electromagnetic induction, the use of high-frequency currents is a distinct advantage.

The experiments here described are merely a preliminary to large scale experiments in actual closed circuit telegraphy the writer hopes to be able to try later on.

One drawback to the magnetic induction form of telegraphy is the rapid rate at which the effect falls off with the distance. In the case of true radiation at long distances the forces vary inversely as the distance, but a more rapid rate of decay, something between the inverse cube and inverse square, holds good for the inductive effect at least at short distances. Hence the use of magnetic oscillators as transmitters is never likely for this reason alone to rival the electric or open oscillator, but there may be circumstances under which it is possible to use them with advantage. In conclusion the author desires to mention that the actual measurements recorded in this paper were taken by his assistant, Mr. G. B. Dyke, B.Sc., with the kind help of Mr. K. W. McMillan, and to these gentlemen is due an acknowledgement of their share in the work, in making these observations with much intelligence and care.

## DISCUSSION.

Mr. A. RUSSELL thanked the author for his clear exposition of the theory of electric oscillators and for the valuable formulæ he gave for magnetic oscillators. The experiments described were exceedingly interesting and instructive. It was, however, difficult to interpret some of the results obtained. He asked whether the author had been able to find a position for the secondary coil in which no current was induced in it. Possibly buried pipes or metallic conductors, iron girders in adjacent buildings or iron railings in the neighbourhood, might have had some influence in modifying the inductive effects between the coils.

Mr. TAYLOR said that the Post Office had tried arrangements similar to those described for magnetic space telegraphy, and had found that the closed circuit method was inferior to that in which earth-connected vertical coils were used. The ratio between the received currents when using an earth-connected and a closed circuit was often as great as eighty to one, and he thought the improvements in the closed-circuit method would have to be very great to make it equivalent to the earthed-circuit method.

## V. *On the Use of Variable Mutual Inductances. By* Albert Campbell, *B.A.** (From the National Physical Laboratory.)

[Plate IV.]

### Contents.

1. Introductory.
2. Mutual versus Self Inductance.
3 & 4. Construction and Adjustment of Variable Mutual Inductances.
5, 6, 7 & 8. Uses of the same for measuring unknown Mutual Inductance, Self Inductance, and Capacity.
9. Appendix. The Scale of the Excentric-Coil Mutual Inductance.

### 1. *Introductory.*

The accurate measurement of small self inductances and capacities has of late years assumed an increased importance owing to its direct practical application in connexion with wireless telegraphy; and much important and interesting work has been done on the subject especially as regards the calculation and measurement of self inductance. For some time past I have found that suitably designed mutual inductances are of the greatest assistance in such measurements, and I incline to think that by their use some of the sources of error inherent in other methods are avoided. The chief method (B) here described was designed for the measurement of the small self inductances (1 to 200 microhenries) used in wave telegraphy, and it has proved most useful in such work. Before I proceed to describe the designs and the methods of use which I have found valuable, let us consider and compare mutual and self inductances in relation to their capabilities in practical use.

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* Read October 25, 1907.

### 2. *Mutual versus Self Inductance.*

For a number of reasons mutual inductances are more easily dealt with than self inductances. Those of invariable value, when properly designed, have the following advantages:—

(*a*) The absolute values can be calculated with much more certainty from the geometrical dimensions, since the formulas for mutual inductance are of high theoretical accuracy while those for self inductance are much less satisfactory *.

(*b*) Unless the conductors are highly stranded, the current distribution varies with frequency and in general the self inductance will also vary. By keeping the two circuits at a relatively large distance from one another the mutual inductance is practically free from this effect.

(*c*) The effects of distributed capacity are probably less in mutual than in self inductances. In all cases the distributed capacity of one of the two coils can be made very small by sufficiently decreasing the number of turns in it (or opening them apart) while increasing the number in the other coil to keep the M constant.

When the mutual inductance is of the variable type, it can always be designed so that its value can be varied continuously from *negative to positive through zero*. This is a very great advantage, for with variable self inductance standards the incapability of reaching a zero value is a distinct drawback.

### 3. *Construction of Variable Mutual Inductances.*

For short range standards the construction used by Lord Rayleigh may be employed †. In this the primary coil is inside the secondary and can be rotated round a common diameter as in the Ayrton-Perry variable self inductances, and the whole positive range is comprised within 90° of turning, which gives a scale not nearly open enough for high accuracy. I have therefore designed a different and more elaborate arrangement which gives a much more extended scale. I proceed to describe the first model, which has been in use for a considerable time with most satisfactory results.

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* For example see Rosa, *Bull. Bur. Stands.* vol. ii. p. 161 (1906); and Strasser, *Ann. der Phys.* vol. xviii. p. 763 (1905).

† Phil. Mag. vol. xxii. pp. 469-500 (1886); Collected Papers, vol. ii. p. 155.

The general arrangement of the apparatus is shown diagrammatically in figs. 1 and 2, which are plan and side view respectively. The primary * circuit consists of two equal coaxial coils C and C′, which are connected in series, their windings being in the same direction of turning. The secondary consists of the coils D and F in series. Of these coils, D is movable, being mounted on an excentric axis Q so as to be free to turn in a plane parallel to those of C and C′ and midway between them. Rigidly connected with the movable coil is a pointer H which moves over a circular scale of about 180° in extent and graduated to read directly. The coil F is subdivided into ten sections, which are in series, each of them being 0·1 millihenry, and their junctions are brought to a set of separate terminals or studs with a turning head. The range of the moving coil extends from $-0{\cdot}002$ to $+0{\cdot}11$ millihenry. This gives a continuous range from 0 up to 1 millihenry, readable near zero to 0·02 microhenry, to 1 part in 500 at 0·1 millihenry, and to 1 in 5000 at 1 millihenry. The subdivision of the coil F is easily carried out by the following artifice. The coil is wound with uniformly stranded wire of ten insulated strands, all the strands are connected in series, and the whole adjusted

Fig. 1.

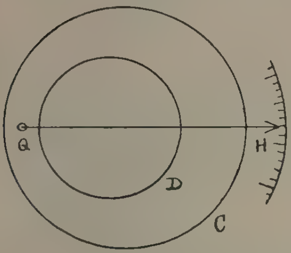
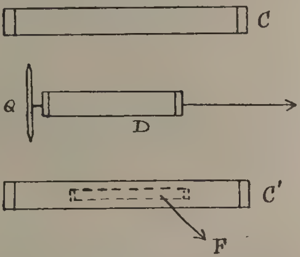


Fig. 2.



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* It is only for convenience of description that the circuits are here distinguished as *primary* and *secondary*. They are really quite interchangeable.

to give 1 millihenry [*]. If the stranding has been properly done, it will be found that no one of the sections differs from its neighbours by more than 1 part in 1000, and each is 0·1 millihenry. The placing of the movable secondary coil midway between the planes of the two primary ones ensures that small axial displacements shall have very little effect on the mutual inductance.

### 4. *Adjustment and Calibration of Variable Mutual Inductances.*

The equality of any pair of sections can be tested by connecting them in series with their windings in opposition in circuit with a ballistic galvanometer and reversing the current in the primary. It should be noticed here that, if a primary coil have any number of secondary circuits, the mutual inductance to all the secondaries in series is equal to the algebraic sum of their separate mutual inductances (+ or − according to the direction of the winding). Owing to this very important property we can build up and step down in the values as easily as if we were dealing with resistances, and there is the further simplification that we can subtract as well as add the values. The marking of the scale and the setting of the coil F are done by comparison with a fixed standard mutual inductance such as I have described elsewhere [†]. The comparison may be made by Maxwell's method [‡], using a sensitive ballistic galvanometer or a vibration galvanometer as detector. When a vibration galvanometer is used as in fig. 3, it should be remembered that, for a balance, *two* conditions must be satisfied, viz.,

$$M_1/M_2=R_1/R_2$$

$$\text{and} \qquad L_1/L_2=R_1/R_2,$$

where $R_1$ and $R_2$ include the resistances of the secondary coils. In order that the second condition may hold, it is necessary to introduce into one of the secondary branches a coil *a* whose self inductance can be continuously varied ; by alternate adjustments of $R_1/R_2$ and the self inductance of this coil, a balance is easily obtained. The fact that $R_1$ and $R_2$ are partly of copper coils is apt to introduce some inaccuracy. The copper resistance, however, can usually be largely swamped without losing too much of the sensitivity.

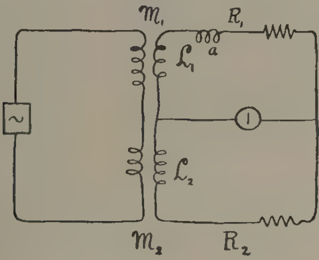
---

[*] Mutual Inductances with stranded wires were used many years ago by Prof. Brillouin (*Thèses presentées à la Faculté des Sciences de Paris*, 1882).

[†] Phys. Soc. May 1907; Phil. Mag. [6] vol. xiv. p. 494, Oct. 1907. Also see Proc. Roy. Soc. A. vol. lxxix. p. 428, June 5, 1907.

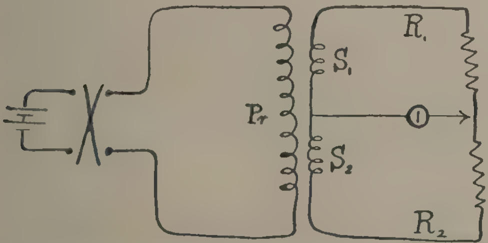
[‡] Maxwell, Elect. and Mag. 2nd edition, ii. § 755.

Fig. 3.



In connexion with this method I may mention that it is perfectly applicable to the case of comparing the mutual inductances between one primary circuit and two separate secondaries. This case is shown in fig. 4.

Fig. 4.



By this means a thorough intercomparison (or adjustment) can be made between the various sections of a subdivided mutual inductance. By further sets of secondary coils for higher multiples (10, 100, and so on) any desired range may be obtained. By stranding (or otherwise subdividing) the primary coils as well as the secondary ones a multiple-range inductance is the result. In all cases the subdivision can be effected by other means than stranding the wires, which is merely used to avoid the separate adjustment of the sections. The main objection to the subdivision by stranding is that it increases the distributed capacity of the coils. With coils of many turns this might be a serious drawback, but with the relatively few turns required for 1 millihenry the effect can only be very small and I have not yet noticed it in practice. It should be kept in mind that, as in all standard inductances, eddy currents should be prevented by using only highly stranded wire for the winding of every section and by avoiding the use of any metal parts near the coils.

By varying the relative dimensions and positions of the two fixed primary coils CC′ (fig. 2) and the movable secondary D, the scale can be considerably modified, and can be designed to suit special purposes. The most generally useful type of scale becomes more and more open as the readings approach zero, for this gives good accuracy of reading even at very low values. Fortunately in the present instance this type of scale can readily be attained. Further discussion of this point will be found in an Appendix.

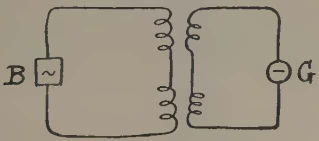
### 5. *Employment of Variable Mutual Inductances.*

Variable inductances of this type may be used for several purposes. Their most obvious application is to the calibration of ballistic galvanometers for magnetic testing; and I have found them of the greatest value for quickly adjusting a galvanometer circuit to read B directly for any given iron ring. I need not dwell on this application at present, but proceed to describe several methods to which they are applicable.

### 6. *Measurement of Mutual Inductance* (Method A).

Any unknown mutual inductance, whose value lies within the range of the variable standard, can be at once determined by connecting the primaries of the unknown and the variable in series to B (fig. 5), a source of alternating or intermittent current, while the secondaries, with their windings in opposition, are connected in series to a vibration galvanometer G. The variable M is then adjusted to bring the deflexion to zero, and the reading gives directly the value of the unknown M. This is an extremely simple method as it involves no knowledge of any resistances. A ballistic galvanometer and a commutated current may be used. This method does not apply to mutual inductances higher than the maximum value of the variable standard; for these another method will be described below (see § 8).

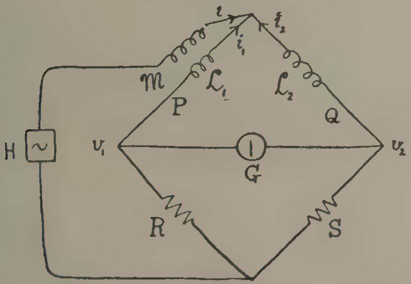
Fig. 5.



### 7. *Measurement of Self Inductance* (Method B).

In fig. 6 let a variable mutual inductance M whose *primary* includes the subdivided coil be connected into a Wheatstone's network, as shown, along with a self inductance $L_2$. Let the resistances of the arms be P, Q, R, and S respectively, the self inductance of the arm P being $L_1$ (the secondary coils of M) and that of Q being $L_2$. Let H be a source of periodic current and G a vibration galvanometer tuned to resonance with it, so that we may take the wave form of the currents to be a sine curve. Let the instantaneous potentials of the three upper corners be $v_1$, 0, and $v_2$ respectively, and

Fig. 6.



the instantaneous values of the currents into the upper corner be $i_1$, $i_2$, and $i$ as marked. Let $p=2\pi n$ where $n$ is the frequency, and for convenience of writing let $p\sqrt{-1}$ be denoted by $\alpha$, so that $\alpha^2=-p^2$. The mutual inductance M may be made positive or negative according to the way in which the coils are connected; and in all that follows we might write $\pm$M for M throughout. When the galvanometer shows a balance, $v_1=v_2$, and the instantaneous value of the current through G is zero.

Also

$$i=-i_1-i_2.$$

Accordingly we may write

$$(P+L_1\alpha)i_1-M\alpha i=(Q+L_2\alpha)i_2\,;$$

therefore

$$[P+(L_1+M)\alpha]i_1=[Q+(L_2-M)\alpha]i_2$$

also

$$Ri_1=Si_2.$$

Hence

$$S[P+(L_1+M)\alpha]=R[Q+(L_2-M)\alpha].$$

Equating the real and imaginary parts each to zero we have

$$SP=QR, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$S(L_1+M)=R(L_2-M). \quad . \quad . \quad . \quad . \quad . \quad . \quad {}^*(2)$$

Exactly the same equations hold when the positions of the source and the galvanometer are interchanged.

The most useful case is when the noninductive arms are made equal, *i. e.* S=R; then (1) and (2) become

$$P=Q,$$

and

$$L_2-L_1=2M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This case gives an extremely convenient way of measuring small self inductances, which is done as follows.

The arrangement is shown in fig. 7. The noninductive arms are equal (R, R). In the arm AB there is the secondary coil $a$ of self inductance L in series with a practically noninductive rheostat *r*. In the arm AC is placed a "balancing" coil *b* also of self inductance L and of resistance equal to or slightly greater than that of *b*. By adjusting *r* the bridge will balance when $M=0$. The small self inductance N to be measured is now inserted in series with coil *b* in arm AC, and a balance obtained by altering *r* and M. Then, by (3), $N=2M$. Thus N is found directly from the reading of M, and the range of values that can be measured runs from 0 up to twice the highest reading of the variable mutual inductance.

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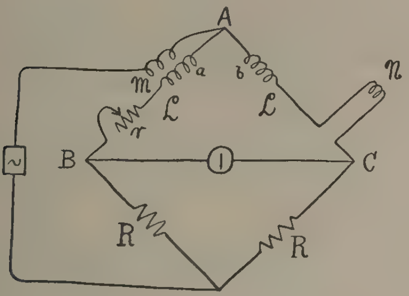
* (Nov. 22, 1907.) I find that a case slightly more complicated than that of Equation (2) has been worked out and applied in an ingenious manner by Grätz (Wied. *Ann.* vol. 50. p. 766, 1893).

Fig. 7.



[For values of N above this range the more general case (equations (1) and (2)) may be used.] The L's of the coils *a* and *b* should be adjusted to equality once for all by putting M at zero and setting one of the coils. An exact setting is convenient, but not necessary, for if $L_a$ and $L_b$ differ slightly, they can be balanced (without N) by a small reading $M_0$. If M be the reading for balance when N is inserted, then $N=2(M-M_0)$*.

Even if *a* and *b* are well matched, it is always well to begin by reading their difference if any.

It will be noticed that the method is really a differential one; when N is introduced into the arm AC no alteration has

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* (Oct. 1st, 1907.) If, instead of the coil N, a condenser K with series (absorption) resistance *q* be added, then

$$1/p^2K=2M,$$

and $q=$change in *r*.

to be made in the other arms except to increase the resistance of AB by an amount equal to the resistance of the coil N. But although it has *all the advantages of differential measurement, the reading can be made to give* N *directly* without having to take a difference at all. This is due to the use of the inductive balancing coil $b$ [*].

The method has the advantage that it does not require the knowledge of the absolute value of any resistance. The noninductive bridge arms must be equal; to check the equality they can be interchanged. For the noninductive adjustable resistance $r$ I usually employ a special rheostat consisting of two slightly flattened thin wires running parallel to one another at a few millimetres distance, with a sliding contact piece across them to complete the circuit. The inductance of such a rheostat can be approximately calculated, and may thus be allowed for when measuring very small self inductances. The inductance of the part added to compensate for the introduction of N has merely to be subtracted from the result.

In practice the method proves very convenient; with the variable mutual inductance described above, self inductances of any value from 0·1 microhenry up to 2000 microhenries can be measured directly without the bridge being altered in any way except in the rheostat $r$. In a later model shown in the accompanying Plate IV. the whole scale of the movable coil corresponds to 20 microhenries, and at this value it can be read to 1 or 2 in 1000—at 200 microhenries to 1 or 2 in 10,000. All the resistances of the coils are very low, and the sensitivity can be considerably increased by using M. Wien's method of connecting the vibration galvanometer to the bridge by means of a transformer of suitable ratio ($n_1/n_2$ small). The method will also give the difference between two unknown self inductances introduced into AB and AC.

The addition of a condenser to Method (B) gives an interesting corroborative method (Method C). If (fig. 8) the condenser K is in shunt with S, it may be shown that, for a balance,

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[*] If coil $b$ be made noninductive we revert to Maxwell's method of comparing the M of a pair of coils with the L of one of them. Equation (2) then reduces to $L/M = -(1+R/S)$. Maxwell, 'Elect. & Mag.' 2nd edit. vol. ii. § 756.

$$SP-RQ=p^2RSK(L_2-M), \qquad (4)$$

and

$$S(L_1-M)-R(L_2+M)=QRSK. \qquad (5)$$

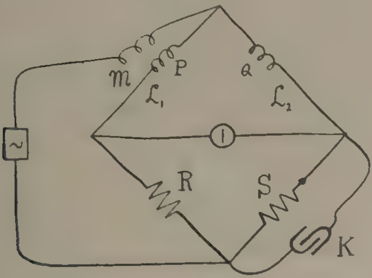
When $R=S$ and $L_1=L_2=L$,

$$P-Q=p^2RK(L-M), \qquad (6)$$

and

$$2M=QRK. \qquad (7)$$

Fig. 8.



Equation (7) gives a check of K against M without having to measure the frequency. The adjustment is dependent on $p$, however, and therefore a steady frequency is necessary, which is not required in Carey Foster's Method.

From (6) and (7) we get

$$Q(P-Q)=2p^2M(L-M). \qquad (8)$$

### 8. *Measurement of Mutual Inductance greater than Standard* (Method D).

When the unknown M is beyond the range of the variable standard the following method is applicable.

Let the primaries of the variable standard and the unknown M be connected up as in fig. 9, adding to the latter if necessary a coil $c$ of sufficient self inductance to make N considerably greater than L. Let a balance be obtained (by Maxwell's Method) and hence

$$PS=RQ, \qquad (9)$$

and

$$NS=LR. \qquad (10)$$

Now, without altering any other branches, let the secondary coils be introduced (in opposition) into the galvanometer circuit as shown in fig. 10.

Fig. 9.

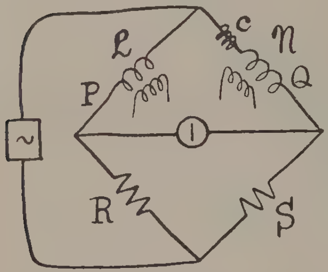
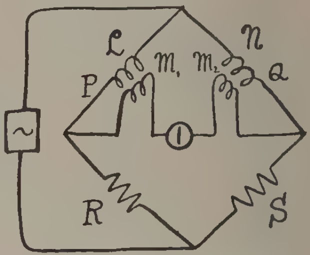


Fig. 10.



If a balance be now obtained by adjusting the variable M, then it is easy to show that

$$(P+R)M_2=(Q+S)M_1, \quad . \quad . \quad . \quad . \quad (11)$$

and $$PS-p^2M_2(L-M_1)=QR-p^2M_1(N-M_2). \quad . \quad (12)$$

Since (9) and (10) still hold, these equations both reduce to

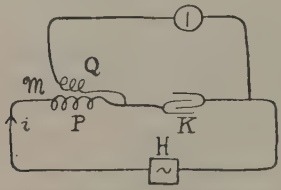
$$RM_2=SM_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

which gives $M_2$ immediately in terms of the reading $M_1$ and the resistances R and S (neither of which include copper coils). The self inductance of the added coil *c* need not be known, but if R and S have not a slide wire between them, *c* ought to be slightly adjustable, which is easily arranged by having part of it a small coil of a few turns that can be shifted about over the remainder.

### 9. *Measurement of Capacities* (Method E).

As in fig. 11, let the primary P of the variable standard

Fig. 11.



be connected in series with a condenser K to a periodic source H, while the secondary Q is connected to a vibration galvanometer as shown. Let M be adjusted until the galvanometer gives no deflexion, in which case we have

$$\pm M\alpha i = \frac{1}{K\alpha} i, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\text{or} \qquad \pm p^2 MK = -1.$$

The secondary must be connected in such a direction that the lower sign is taken, and thus

$$p^2 MK = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where M is in henries and K in farads.

It is interesting to notice that this formula is similar in form to that for resonance with a condenser and self inductance, viz. $p^2LK=1$. With self inductance we usually tune to get a maximum current through the galvanometer; with mutual inductance, on the other hand, we tune for zero current. If M is in henries and K in microfarads, equation (15) gives

$$n \doteqdot \frac{159 \cdot 2}{\sqrt{MK}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and if M and K are known, the frequency can be determined.

If the frequency is not fairly high, the product MK becomes inconveniently high, *e. g.* M = 1 henry, K = 10 mfds. gives $n \doteqdot 50 \sim$ per sec. In contrast, $n \doteqdot 10{,}000$ for 256 microhenries and 1 mfd.

The following is a practical example of the method:—The source was a microphone hummer for $2000 \sim$ per sec. and the detecting instrument a tuned telephone. A balance was obtained for $K = 10 \cdot 06$ mfds. and $M = 0 \cdot 626_6$ millihenry, whence the frequency was found to be $2004 \sim$ per sec. With another condenser (a carefully calibrated subdivided one), capacities of 1·000, 0·900, 0·800, 0·701, 0·601, and 0·551 gave

$n = 1999$, 2005, 2009, 2001, 2004, and $2001 \sim$ per sec.

respectively; the variable M was not, however, so accurately readable in this case.

In this method it is very important that the insulation resistance S of the condenser shall be high. If the leakage is not negligible, equation (14) becomes

$$\pm M\alpha i = \frac{S}{1+SK\alpha} i$$

$$= \frac{S(1-SK\alpha)}{1+p^2S^2K^2} i. \quad . \quad . \quad . \quad . \quad (17)$$

Putting the real part equal to zero we have

$$1/(1+p^2S^2K^2)=0,$$

showing that an exact balance can only occur when $p^2S^2K^2=\infty$. In practice the insulation resistance of a good mica condenser is quite high enough to give a good balance; even a good paraffined paper condenser will work fairly well, but an ordinary one gives only a minimum deflexion on the galvanometer *.

The application of this null method to the high frequencies used in wave telegraphy is made difficult by reason of the coexistence of two frequencies in the waves used.

## APPENDIX.

### *Scale of Excentric-Coil Variable Mutual Inductance.*

The problem of finding the mutual inductance between two circles (thin circular coils) not in the same plane and whose axes are parallel has not yet been solved, as far as I know. For any particular case the result may be calculated by quadratures as follows.

In fig. 12 let the two circles be H and G, of radii $a$ and A respectively, $b$ being the distance between their planes, and $q$ the distance between their axes. Let the circle H′ be the projection of H on the plane of G, and let BCD be an elementary strip concentric with H′ of radial width $\partial r$, where $O'B=r$. Let $CO'B=\theta$. Then the M between circle H and strip BCD

$$= \frac{\theta}{\pi}. \text{(M of circle H and annulus TBD)}$$

$$= \frac{\theta}{\pi}\cdot\frac{\partial}{\partial r}\left[-4\pi n_1 n_2 \sqrt{ar}\left\{\left(k-\frac{2}{k}\right)F_1+\frac{2}{k}E_1\right\}10^{-9}\right]\partial r,$$

---

* This may, however, be due more to absorption than leakage.

or

$$\partial M = \frac{\theta}{\pi} \cdot \frac{2\pi n_1 n_2 10^{-9}}{\sqrt{(a+r)^2+b^2}} \left\{ 2a(F_1 - E_1) - (r-a)\left(\frac{k}{k'}\right)^2 E_1 \right\} \partial r, \quad . \quad . \quad . \quad (18)$$

where

$$k = 2\sqrt{ar} / \sqrt{(a+r)^2+b^2} = \sin\gamma,$$

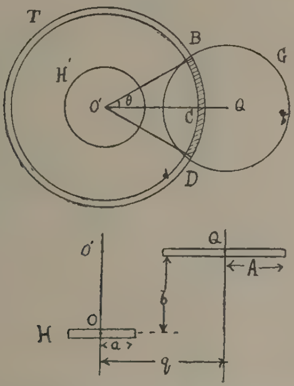
$$k' = \cos\gamma,$$

and $F_1$ and $E_1$ are complete elliptic integrals to modulus $k$.

Also

$$\theta = \cos^{-1} \frac{r^2+q-A}{2rq}.$$

Fig. 12.

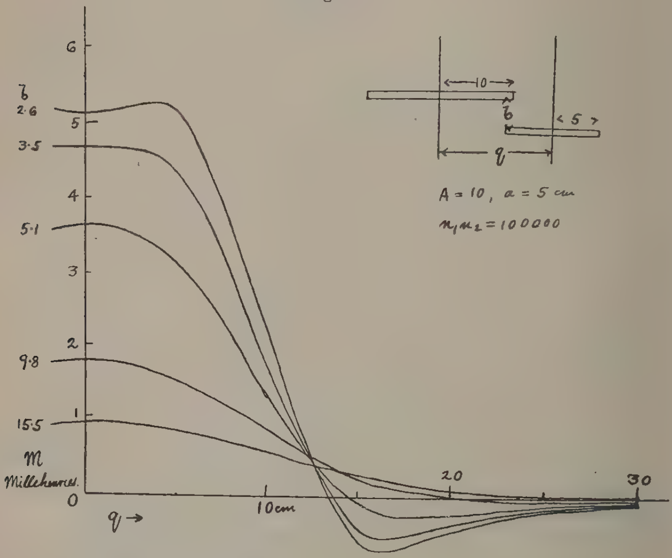


Thus when $a$, A, $b$, and $q$ are given, by calculating $\partial M$ by (18) for a series of equally spaced successive values of $r$, from $r = q \sim A$ to $q + A$, and adding in the usual way, the value of M can be approximately found. By this process, which is rather laborious, the following approximate values were found, when $A = 20$, $a = 10$, $b = 5$ cm and $n_1 n_2 = 100{,}000$:—

| $q$. cm. | M. millihenries. |
|---|---|
| 0 | 9·3 |
| 5 | 9·7 |
| 10 | 9·9 |
| 15 | 6·7 |

It is interesting to notice that the maximum value of M does *not* occur when the axes of the coils are coincident ($q=0$). To avoid the labour of the calculation by quadratures, some rough experiments were made with coils of finite section (about $0·6 \times 0·6$ sq. cm.). From one set of these the series of curves shown in fig. 13 was obtained. The coils were of mean radii $A=10$, $a=5$ cm. respectively, $n_1 n_2$ being 100,000, and $q$ and $b$ were given various values. It will be seen that for $b=2·6$ the maximum M does not occur

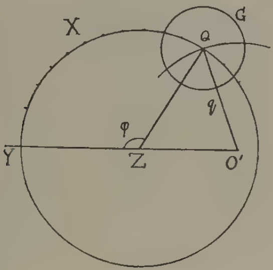
Fig. 13.



at the symmetrical position (just as in the case calculated, above). For equal coils, however, the coaxial position gives the maximum M. The curves also show that the value of $q$ for which M vanishes and changes sign becomes greater and greater as the mean planes of the coils are set at greater distance from one another. The ratio of $A/a$ has been made 2 in this example, as that is a suitable proportion for use in a practical instrument. Having now the curves connecting M with $q$, the distance between the axes of the two coils, it is easy to investigate the variety of angular scales obtainable by turning one of the coils round an axis parallel to its own axis.

Let a definite value of $b$ be taken, say $b=5$, which is convenient in practice, and in fig. 14 let the smaller coil be movable about an axis perpendicular to the plane of the paper at Z, and let the radius $ZQ=\rho$. Let the distance $ZO'$ of the axis Z from $O'$ (the point where the axis of the other coil cuts the plane of the paper) be $h$. Let the angular scale reading $YZQ=\phi$.

Fig. 14.

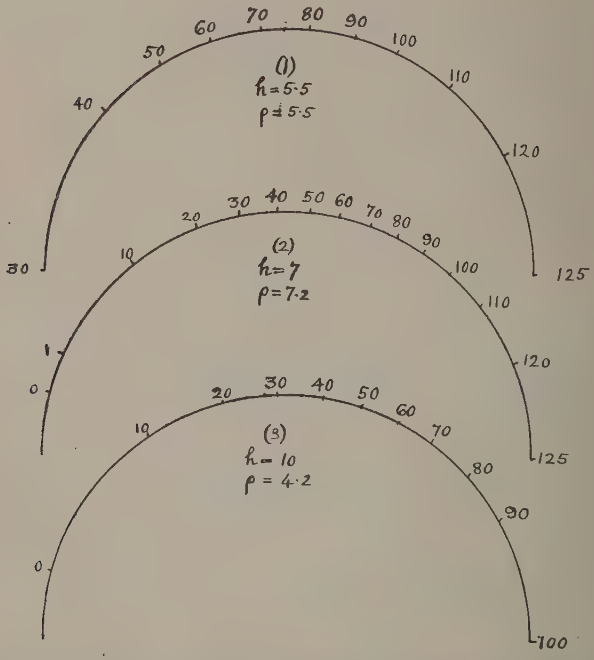


Then by choosing various values for $\rho$ and $h$ we can get a wide variety of movement of the centre Q (of the movable coil) relatively to $O'$, and hence a variety of scale calibrations. Since $\cos\phi=\dfrac{q^2-h^2-\rho^2}{2h\rho}$, the angle $\phi$ corresponding to a reading M can be got from $q$, $h$, and $\rho$. The scale can be most easily set off by the following geometrical construction. Draw a circle with centre Z and radius $\rho$, and make $ZO'$ equal to $h$. Then find on the $(b=5)$ curve in fig. 13 the $q$ corresponding to a particular value of M. With this value of $q$ as radius, a circle with centre $O'$ will cut the circle X at the point of the scale required. By taking successive values of M the direct-reading scale can be set off round the circumference of the circle X.

As an example, in fig. 15 are given three scales for $h=5{\cdot}5$, $\rho=5{\cdot}5$; $h=7$, $\rho=7{\cdot}2$; and $h=10$, $\rho=4{\cdot}2$ respectively, when $A=10$, $a=5$, $b=5$, and $n_1\,n_2 \doteqdot 33{,}000$.

Although these scales give very different ranges, the general characteristic of all three is an openness at both ends; no proportions that I have tried tend much to make the scale uniform throughout. The openness at the lower values is a distinct advantage, however, when dealing with small inductances; but care must be taken, when marking that part of the scale, that the subdivisions shall be accurately marked and not put in merely by interpolation.

Fig. 15.



Scale (2) in fig. 15 is the best of the three and is nearly that which I have used in practice.

In conclusion I desire to express my best thanks to Dr. Glazebrook, for his kind encouragement and valued advice.

Sept 21, 1907.

## Discussion.

Dr. **Fleming** asked the author how his method compared in accuracy with the Carey Foster method, using a commutator to add up the impulses. He had frequently measured mutual inductance by determining the self-inductances of the primary and secondary joined in series, with the windings first helping each other and then opposing each other. He asked if the author's methods were applicable when dealing with the high frequencies used in wireless telegraphy.

Mr. A. **Russell** agreed with the author in thinking that mutual inductance was sometimes easier to calculate and easier to measure than self-inductance. The variable mutual inductance standard shown was very compact, and would be a great acquisition to a physical laboratory. In calculating self-inductance the variable current density over the cross-section of the wire sometimes brought in serious difficulties. The measurement of self-inductance in these cases might possibly be made by Mr. Campbell's method, but the limitations introduced into his formulæ by the assumption that the back E.M.F. was proportional to the rate at which the total current was changing would have to be considered.

Mr. **Campbell**, in reply to Dr. Fleming, said the methods of measuring mutual inductance described in the paper were more convenient and direct reading than that of Carey Foster, although they were not so absolute. They were of higher accuracy than the difference method, in which the primary and secondary were put in series and their combined self-inductance measured both with the windings helping and opposing. In reply to Dr. Fleming and Mr. Russell, the methods of measurement described were not intended, in general, for use with high frequencies of the order of those used in wave telegraphy, where resonance methods appear to be the only ones available. Mr. Campbell mentioned that the correction for the small inductance of the rheostat in method (B) could be got rid of by making the rheostat of constant inductance, this being attained by running a manganin and a copper wire side by side with a travelling contact across them, the current entering by the copper at one end and leaving by the manganin at the other.

# VI. *A Freehand Graphic way of determining Stream Lines and Equipotentials.* *By* L. F. RICHARDSON *.

[Plate V.]

## SCHEME OF PAPER.

I. On the need for new methods.
II. The first idea of freehand solution and confirmation of its accuracy.
III. The conditions which the solution of $\nabla^2 V = 0$ must satisfy in order that it may be determinable by a single graph.
    (*a*) When the guiding lines are normal to a family of surfaces. Possible types—test cases.
    (*b*) Thin shells.
    (*c*) Screw symmetry—example.
IV. Points and lines of equilibrium.
V. Equations other than Laplace's—variable conductivity.
VI. Boundary conditions.
VII. Miscellaneous notes on draughtsmanship.
VIII. Estimation of errors.

## I. *The Need for New Methods.*

THE Laplacian differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

has received an extraordinary amount of attention during the last century owing to the great number of physical quantities, the space distribution of which can be determined from its integrals. The analytical integrals hitherto obtained by such means as Fourier series, Bessel functions, spherical and other harmonics make it possible to determine the distribution when the boundary conditions bear relation to certain simple types of surface, such as parallelipipeds, cylinders, spheres, ellipsoids, anchor rings, &c.

Now for physical research this is well enough. It is usually possible to arrange the instruments so that the parts involved are of these simple forms. The wires may be wound in circular rings of small cross-section, as in Helmholtz's galvanometer. The pieces of substance for the measurement of specific properties may be shaped into square bars, as in Forbes's experiments on the flow of heat. Or, as in Kelvin's attracted disk electrometer, parallel plates may be made practically infinite by his device of the guard-ring.

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* Read November 8, 1907.

But for the purposes of the engineer this is of very limited application. If he is to handle partial differential equations freely, they must be applicable to bodies of most various shapes, such, for example, as the toothed core-plates of dynamos, the water surrounding ship shells and screw propellers, the space between turbine blades, and a host of other forms, too irregular to be readily described.

Further than this, the method of solution must be easier to become skilled in than are the usual methods with harmonic functions. Few have time to spend in learning their mysteries. And the results must be easy to verify—much easier than is the case with a complicated piece of algebra. Moreover, the time required to arrive at the desired result by analytical methods cannot be foreseen with any certainty. It may come out in a morning, it may be unfinished at the end of a month. It is no wonder that the practical engineer is shy of anything so risky.

Harmonic functions have, however, one very strong point in comparison to the methods put forward in this paper, and that is their accuracy. Once we have determined V as an infinite series of harmonic functions, it is usually not much more labour to obtain an accuracy of 1 in a million than of 1 in ten.

Now it is true that in the determination of absolute electric standards measurements are made to 1 in 100,000 or to an even greater refinement. But for most chemical and physical work 1 in 1000 is more like the limit attained. And in any new branch of research, two, five, or even ten per cent. are very welcome. The root of the matter is that the greatest stimulus of scientific discovery are its practical applications. And here, in the design of machinery for example, cost rules everything, and this can seldom be foreseen as near as 1 per cent.

To sum up. The existing methods of solving Laplace's equation are susceptible of great accuracy, but they are slow and uncertain in time and, most serious of all, they can only be applied to very special boundary conditions. There is obviously a demand for a method of solving that group of partial differential equations — of which we may regard Laplace's as the simplest type—which shall, if necessary, sacrifice accuracy above 1 per cent., to rapidity, freedom from the danger of large blunders, and applicability to more various forms of boundary surface.

## II. *The First Idea of Freehand Solution.*

The real simplicity of the space distributions of electric and magnetic phenomena,—so much disguised in the algebraic integrals of the differential equations, but rescued from confusion and clearly set forth by the vector analysts, Heaviside, Walker and others,—leads one to hope for equally simple methods of calculating their numerical values with reference to any boundary whatever.

The beautiful figures of stream and equipotential surfaces published by Maxwell, Lamb and others as the result of harmonic analysis, and by Hele Shaw as the result of experiment, suggest that by imitating their characteristic properties freehand we may, in some small part, attain the result desired.

Maxwell in § 92 of his 'Elementary Treatise on Electricity and Magnetism' speaks of tentative methods of altering known solutions of the Laplacian equation by drawing diagrams on paper and selecting the least improbable. The object of the present thesis is to point out that this method can do far more than merely alter known results, and that it may be so far from tentative as to yield an accuracy of one per cent of the range.

This method of treating potentials, although still far from combining all desirable qualities, and suffering from the restriction to certain types of symmetry, yet from its great freedom within those types may, it is hoped, supply to a certain extent the demand we have indicated.

On turning to Maxwell's figures and picking out those in which V is independent of $z$ so that we have

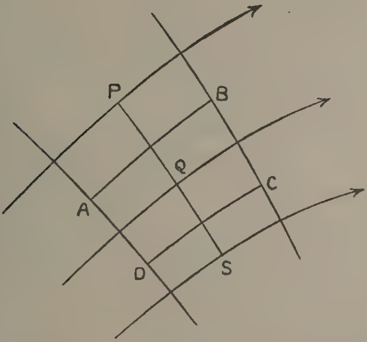
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

it will be seen that while the curves are of the most various shapes yet the chequerwork of all the diagrams possesses these two properties in common:—(1) the corners are orthogonal, (2) when the chequers are small enough the ratio of their length to breadth is the same in all parts of the field.

The proof of this follows most conveniently from the consideration of the motion of a liquid when the lines of flow lie in parallel planes and the motion is the same at all points of any normal to these planes. Draw three adjacent stream lines defining two adjacent tubes of flow.

Take two points A and B on the mid line of one tube, and from A and B draw normals to the direction of flow cutting the mid line of the other tube in D and C respectively.

Fig. 1.



Halfway between AD and BC draw a line PQS normal to the direction of flow so that PQ is the width of one tube and QS of the other. Now if the fluid is incompressible and we have drawn the tubes so that the flow in each is the same, then the respective velocities are to one another inversely as PQ and QS. Let the velocities be $\frac{k}{\mathrm{PQ}}$ and $\frac{k}{\mathrm{QS}}$. Next let us take the line integral of the velocity round the small rectangle ABCD. The sides AD and CB are normal to the flow and so contribute nothing. The sides AB and CD contribute

$$\mathrm{AB}\times\frac{k}{\mathrm{PQ}}-\frac{\mathrm{DC}\times k}{\mathrm{QS}}=k\left\{\frac{\mathrm{AB}}{\mathrm{PQ}}-\frac{\mathrm{DC}}{\mathrm{QS}}\right\}.$$

Now $\frac{\mathrm{AB}}{\mathrm{PQ}}$ is the ratio of the length along the flow to the breadth across the flow of the small chequer which has A, Q, B, P, at the mid points of its four sides. It will be convenient to have a special name for this quantity, and I propose to call it the "chequer ratio" with the understanding that length along the flow is always in the numerator, and that the chequer is so small that its size no longer causes an appreciable deviation from the accuracy obtained by using infinitesimals. Then we have :—

$$\text{Line integral of the velocity around ABCD}=k\times\left\{\begin{array}{l}\text{Difference between successive}\\ \text{chequer ratios in a direction}\\ \text{perpendicularly across flow.}\end{array}\right.$$

Now the curl of the vector velocity is defined as the line integral round a small circuit divided by the area of that circuit—that is in this case by the area ABCD which will in the limit be equal to the mean of the areas of the two adjacent chequers. So that we have :—

$$\text{curl of the velocity}=k\,\frac{\begin{array}{c}\text{difference of successive chequer ratios in a direction}\\ \text{perpendicular to velocity}\end{array}}{\text{mean chequer area}}.$$

If the velocity has no curl the chequer ratio must not vary along any line normal to the flow. It may vary from one normal to another, but if on the other hand we prefer to make it constant all over the field, then at any point the distance between successive normals will be inversely as the flow, so that these normals will be contours drawn at equal intervals of a velocity potential.

To return : since the fluid is incompressible the condition for the existence of a stream function is satisfied, and since the stream-lines are drawn so that the flow between each successive pair is the same, it follows that these stream-lines are the contours drawn at equal intervals of a stream function $\psi$. Now it is proved in works on Hydrodynamics that $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}$ is equal to the curl of the vector velocity.

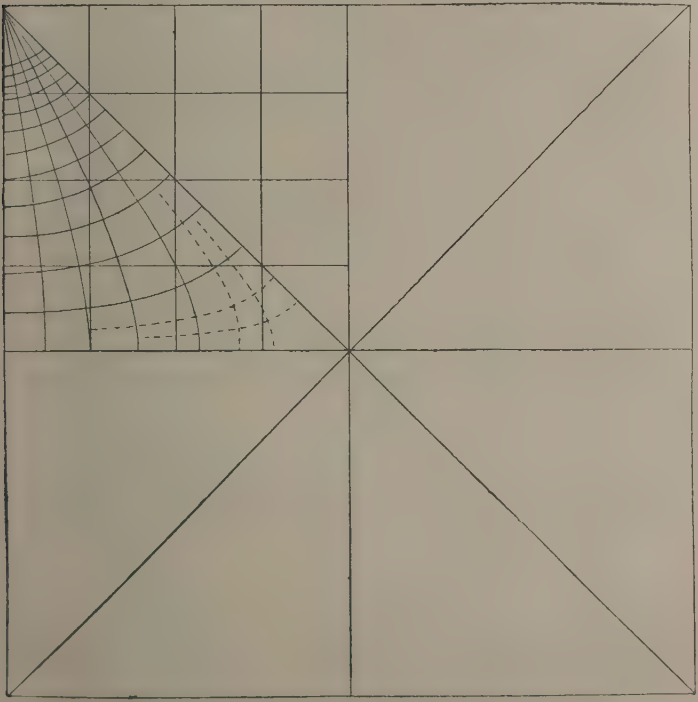
Therefore :—

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=k\,\frac{\text{difference of successive chequer ratios in a direction perpendicular to the contours of }\psi}{\text{mean chequer area}}.$$

And since $\psi$ may be any one-valued function of position on the plane, it is seen that all hydrokinetical considerations have been eliminated from the above equation, which is purely a proposition in differential geometry. The only implication being that contours are drawn at equal intervals of $\psi$ whatever be its physical meaning.

To draw chequers freehand so as to satisfy a difference relation of this sort between the chequer ratios is likely to be toilsome, and we will here consider only the case when $\nabla^2\psi=0$.

Fig. 2.



Supposing then that a chequerwork has been obtained in which the chequer ratio is everywhere the same and in which the given boundary conditions are satisfied, then by the uniqueness of the solution of $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ this chequerwork gives us what we want.

It remains to be shown what accuracy may be expected from the freehand method. This is of course largely a personal matter. I exhibit my own handiwork. Others will doubtless obtain greater precision. Throughout this paper I have chosen test cases in which the analytical verification should not be too difficult. Hence the diagrams look rather stiff and formal and do not in any way do justice to the freedom of this graphic method.

*Example of the solution of* $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.—Along one pair of opposite edges of a square $V = 1$, along the other pair $V = 0$. Find V at all points inside. By symmetry V will be equal to $\frac{1}{2}$ along the diagonals. And again by symmetry, the lines joining mid-points of opposite sides will be normal to the contours of V. So that it is only necessary to find V in half one-quarter of the square. Further, by symmetry, at a corner of the square the contours drawn at equal intervals of V must make equal angles with one another. One starts then by ruling out an accurate square, putting in the diagonals, joining the mid-points of its sides and setting off the equal angles with a protractor. It is convenient to divide the range of V into ten equal parts. Having thus prepared the paper, lines were sketched and amended until further improvement became very slow. The pencil-lines were then firmly fixed in ink. Coordinate lines were drawn in and the values of V at six points were read from the diagram and are given in parenthesis in the accompanying table. The whole work from the beginning of the drawing took two or three hours.

| | | | | |
|---|---|---|---|---|
| | | | | ·5 |
| | | | ·5 | (·47)<br>·466 |
| 1·0 | | ·5 | (·40)<br>·396 | (·365)<br>·364 |
| 1·0 | ·5 | (·307)<br>·300 | (·23)<br>·223 | (·20)<br>·202 |
| ·5 | 0 | 0 | 0 | 0 |

Not until this had been done did I look up the correct values which had been computed from the analytical solution

$$V=\frac{4}{\Pi}\sum_{m\text{ odd}}(-1)^{\frac{m-1}{2}}\frac{1}{m}\operatorname{sech}\frac{m\Pi}{2}\cos mx\,.\cosh mz.$$

These are given in the table beside the numbers read from the graph. From these we find the errors +·007, +·007, −·002, +·004, +·001, +·004. Treating these as all of the same sign, their mean is ·0042.

The error of a graph may well be compared with the total range of V within which the determination was made freehand. In this case the range was 0·5 so that the mean error was 0·84 per cent. of the range, a degree of accuracy which would be sufficient for many purposes.

Having shown that the freehand method is a practical one in a plane, it will be well next to enquire to what types of symmetry it may be extended.

## III. *Possible Types of Symmetry.*

Inasmuch as a single chequerwork is to determine the distribution in the whole of the space considered, we are confined to two coordinates. The freehand method at present offers nothing to compete with the analytic forms in which three coordinates appear, such as:

$$V=\sum_{m,n}(A_m \sin mx+B_m \cos mx)(C_n \sin ny+D_n \cos ny)(Fe^{-\sqrt{m^2+n^2}.z}+Ge^{+\sqrt{m^2+n^2}.}$$

$$V=\sum_{m,k} e^{\pm kz}(A_m \cos m\phi+B_m \sin m\phi)J_m(kr),$$

(Whittaker, 'Modern Analysis,' p. 318)

$$V=\sum_{n,m} C_m r^m (A_n \cos n\phi+B_n \sin n\phi)(\sin\theta)^n.\frac{d^n P_m \mu}{d\mu^n},$$

(Byerly, 'Fourier's Series,' p. 196)

or others like them.

The expressibility of V in terms of two coordinates implies that V is constant along a certain family of lines in space, namely, the intersections of the surfaces over which the said two coordinates are respectively constant. Any particular type of symmetry is most conveniently distinguished by specifying the family of lines along each of which V must be constant.

As it will frequently be necessary to refer to these lines and to distinguish them from the normals to the surfaces V=constant, I propose to call them the "guiding lines."

It is indeed conceivable that by adding together several space distributions in each of which V is constant along a different family of lines, we might attempt the solution of problems which it may be impossible to treat by two coordinates directly, such for example as the motion of a perfect fluid past a three-bladed screw-propeller, or the electrostatic field due to a ring of electrons. It may even be possible to treat the most general distributions by means of sections of the potential surfaces drawn on the leaves of a book of tracing-paper. But these extensions must be left to those who desire the results. This paper deals only with two coordinates.

*Case* (*a*). *The guiding lines are everywhere orthogonal to a family of surfaces.*—Let these be the surfaces over each of which $\gamma=F_3(x, y, z)$ is constant.

Then choosing a particular surface, say $\gamma=\gamma_0$, we wish to draw thereon a chequerwork of orthogonal lines, and we wish this chequerwork, by the motion of each point of it along the guiding line at that point, to sweep out two families of surfaces in space, in such a way that one family may be equipotentials and the other stream-surfaces. This requires that these two families, which we may denote by

$$\alpha=F_1(x, y, z)=\text{const.}, \qquad \beta=F_3(x, y, z)=\text{const.},$$

should be everywhere orthogonal. Therefore the surfaces $\alpha$. $\beta$, $\gamma$ are mutual orthogonal, and consequently the surfaces $\gamma$ must satisfy the condition necessary in any member of a triply orthogonal system (Salmon, 'Geometry of Three Dimensions,' 4th ed. §§ 476 to 486).

But more than this. For we wish to be unrestricted as to the direction of the orthogonal traces of $\alpha$ and $\beta$ drawn upon the surfaces $\gamma$. Therefore, since three mutually orthogonal surfaces necessarily intersect in their lines of curvature (*loc. cit.* § 304), it follows that at every point of the surfaces $\gamma$ there are lines of curvature in every direction. The only form which possesses this property is the sphere or its limit the plane. Therefore the surfaces $\gamma$ are either spheres or planes. This is necessary. We have not proved that it is sufficient. As frequent reference will be made to the theorems proved in Lamé's *Leçons sur les coordonnées curvilignes*, it will be convenient to employ expressions such as (Lamé, § xi. 15) to indicate equation 15 of § xi. of this treatise. The relation of our notation to Lamé's is that his $\rho$ $\rho_1$ $\rho_2$ are replaced by $\alpha$ $\beta$ $\gamma$ and that $\alpha$ $\beta$ $\gamma$ are used as subscripts respectively instead of absence of subscript, 1 and 2. Otherwise the notations are the same.

In particular, if F is any function of position, we will denote by $H_F$ the quantity

$$\frac{1}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2}}$$

which is the reciprocal of the space-rate of F along the normal to the surface F=constant.

Consider the lamina bounded by the two spheres $\gamma$ and $\gamma+\delta\gamma$. The thickness of the lamina is $H_\gamma\delta\gamma$. If $\alpha=V=$the potential, then the surfaces $\beta=$constant are surfaces of flow. Denoting in like manner the distances between two adjacent surfaces of these families by $H_\alpha\delta\alpha$, $H_\beta\delta\beta$, we see that $H_\alpha\delta\alpha$

and $H_\beta \delta\beta$ are the length and breadth of a chequer traced on the surface $\gamma$.

A tube of flow is bounded by the four surfaces $\beta$, $\beta+\delta\beta$, $\gamma$, $\gamma+\delta\gamma$. And its cross section is therefore $H_\beta \,.\, H_\gamma \,.\, \delta\beta \,.\, \delta\gamma$.

Now if the flux has no divergence, then along a tube of flow magnitude of flux multiplied by cross-section = constant. But the magnitude of flux is equal to the negative space-rate of the potential $\alpha$ along the line of flow, and this is $-\frac{1}{H_\alpha}$. Therefore along a line of flow $\frac{H_\beta \,.\, H_\gamma}{H_\alpha} \,.\, \delta\beta \,.\, \delta\gamma$ must be constant in order that the vector space-rate of the scalar $\alpha$—the Hamiltonian $\nabla\alpha$—shall have no divergence. Or equivalently the condition that

$$\nabla^2\alpha = 0 \text{ is } \frac{\partial}{\partial\alpha}\left(\frac{H_\beta \,.\, H_\gamma}{H_\alpha}\right) = 0.$$

Now we may by freehand trial and amendment so arrange the orthogonal lines on the surface $\gamma_0$ that the above relation shall hold true on the surface $\gamma_0$; but we must further enquire what conditions the spheres $\gamma$ must satisfy in order that $\frac{\partial}{\partial\alpha}\left(\frac{H_\beta \,.\, H_\gamma}{H_\alpha}\right) = 0$ shall be true for all values of $\gamma$ when it is true for one $\gamma_0$; and this moreover when $\alpha$ and $\beta$ are otherwise undetermined.

At this stage the fact that the surfaces $\gamma$ are spheres makes a remarkable simplification. For supposing for a moment that they did not possess this property and that $r^\alpha_\gamma$ and $r^\beta_\gamma$ were their principal radii of curvature at any point, then by (Lamé, § xxx. 24)

$$\frac{1}{r^\alpha_\gamma} = \frac{h_\gamma}{h_\alpha} \,.\, \frac{\partial h_\alpha}{\partial\gamma}, \qquad \frac{1}{r^\beta_\gamma} = \frac{h_\gamma}{h_\beta} \,.\, \frac{\partial h_\beta}{\partial\gamma},$$

where $h_\alpha = \frac{1}{H_\alpha}$, and similarly for $\beta$ and $\gamma$.

Equating the two curvatures,

$$h_\beta \,.\, \frac{\partial h_\alpha}{\partial\gamma} = h_\alpha \frac{\partial h_\beta}{\partial\gamma}.$$

Now

$$\frac{\partial}{\partial\gamma}\left(\frac{H_\beta}{H_\alpha}\right) = \frac{\partial}{\partial\gamma}\left(\frac{h_\alpha}{h_\beta}\right) = \frac{-h_\alpha \frac{\partial h_\beta}{\partial\gamma} + h_\beta \frac{\partial h_\alpha}{\partial\gamma}}{h_\beta^2}$$

and therefore vanishes; so that the chequer ratio $\frac{H_\alpha}{H_\beta}$ is a function of $\alpha$ and $\beta$ only.

As has already been stated, to make $\nabla^2\alpha=0$ on the surface $\gamma_0$ we must have

$$\frac{\partial}{\partial\alpha}\left(\frac{H_\beta H_{\gamma_0}}{H_\alpha}\right)=0,$$

which may conveniently be arranged by making

$$\frac{H_\alpha}{H_\beta}=k_0 H_{\gamma_0}$$

where $k_0$ is a constant, so that the chequer ratio is given as a function of position on $\gamma_0$. This is more than sufficient in that it makes

$$\frac{\partial}{\partial\beta}\left(\frac{H_\beta H_{\gamma_0}}{H_\alpha}\right)$$

vanish as well, but the loss of generality involved is found not to matter, while the simplicity gained is a great convenience. Next, because $\frac{H_\alpha}{H_\beta}$ is independent of $\gamma$ it follows that on any other surface $\gamma_1$ we still have

$$\frac{H_\alpha}{H_\beta}=k_0 H_{\gamma_0}.$$

But if $\nabla^2\alpha=0$ is to be satisfied on this second surface we must there have

$$\frac{H_\alpha}{H_\beta}=k_1 H_\gamma,$$

where $k_1$ is a second constant. Therefore regarding $\gamma_0$ as fixed and $\gamma_1=\gamma$ as movable we have $H_\gamma=H_{\gamma_0}\times$ a function of $\gamma$ only.

But $H_{\gamma_0}$ is a function of $\alpha$ and $\beta$ only.

This relation is equivalently expressed by the two equations

$$\frac{\partial^2}{\partial\gamma\partial\alpha}(\log H_\gamma)=0, \qquad \frac{\partial^2}{\partial\gamma\partial\beta}(\log H_\gamma)=0,$$

whence by (Lamé, § xxx. 24)

$$\frac{\partial}{\partial\gamma}\left(\frac{1}{h_\alpha r^\gamma_\alpha}\right)=0, \qquad \frac{\partial}{\partial\gamma}\left(\frac{1}{h_\beta r^\gamma_\beta}\right)=0.$$

But if the surfaces $\gamma$ are planes, then by the equations

(Lamé, § xxx. 24) already quoted

$$\frac{\partial}{\partial\gamma}\left(\frac{1}{h_\alpha}\right)=0=\frac{\partial}{\partial\gamma}\left(\frac{1}{h_\beta}\right)$$

and consequently $\frac{1}{r_\alpha{}^\gamma}$ and $\frac{1}{r_\beta{}^\gamma}$ are independent of $\gamma$. But it is shown by Lamé (§ xxxviii.) that the curvature of the arc of intersection of the surfaces $\alpha$ and $\beta$ is equal to

$$\sqrt{\left(\frac{1}{r^\gamma_\alpha}\right)^2+\left(\frac{1}{r^\gamma_\beta}\right)^2}.$$

So that if the radius of curvature of this arc be $p$ then $p$ is independent of $\gamma$. But $p$ is equal to the length of the normal from the point considered onto the line of ultimate intersection of two consecutive planes of the family $\gamma$ which pass one on either side of the point considered. As the plane moves this length must remain constant. And as this is to be true for every point in space, it is easy to see that if the surfaces $\gamma$ are planes they must intersect in a common axis. We have in this case symmetry about an axis. Or if the axis be at an infinite distance, the planes are parallel, and we have V independent of one of the Cartesian coordinates $x$, $y$, $z$. But if, quite generally, the surfaces $\gamma$ are spheres we have only $\frac{h_\alpha}{h_\beta}$ independent of $\gamma$ and therefore $\frac{r^\gamma_\alpha}{r^\gamma_\beta}$ independent of $\gamma$. If the centres of the spheres $\gamma$ lie in a straight line, then since the orthogonal traces of the surfaces $\alpha$ and $\beta$ on a sphere $\gamma$ may turn round anyhow, we may choose for $\beta$ the planes intersecting in the line of the centres of the spheres $\gamma$. Then $r^\gamma_\beta=0$, and consequently $r^\gamma_\alpha$ is independent of $\gamma$ so that the traces of $\alpha=$const. on the planes $\beta$ are circles. This is the system of toroidal coordinates which has been treated by Professor Hicks in Phil. Trans. 1881, Part II. Now the above reasoning would lead us to expect in these a type of symmetry which can be dealt with by two coordinates—other than symmetry about an axis. But on referring to Hicks's formulæ it is easy to show that this is not possible, for if V be made independent of either of those two of his coordinates which determine position in a plane passing through the axis, then the other of these two will not divide out of the equation $\nabla^2 V=0$, so that all three coordinates must still be present in the integrals. Clearly then, our deductions, though necessary, are not sufficient. I have little doubt that the omission lies in this: that to leave us unrestricted as to the direction of the orthogonal traces of $\alpha$ and $\beta$ upon the surfaces $\gamma$, it is not sufficient that the surfaces $\gamma$ should be spheres. For the curves normal to $\gamma$ which we have called the "guiding lines" must be such that they form one set of lines of curvature of any surface whatever passing through them. To satisfy this condition it seems likely that except when the radius of the spheres $\gamma$ is infinite, the guiding lines will have to be straight and the spheres concentric. This is the symmetry when V is independent of the radius in spherical coordinates, but may vary anyhow with the latitude and longitude.

The only case remaining uninvestigated is that in which the surfaces $\gamma$ are spheres with centres which do not lie on a straight line.

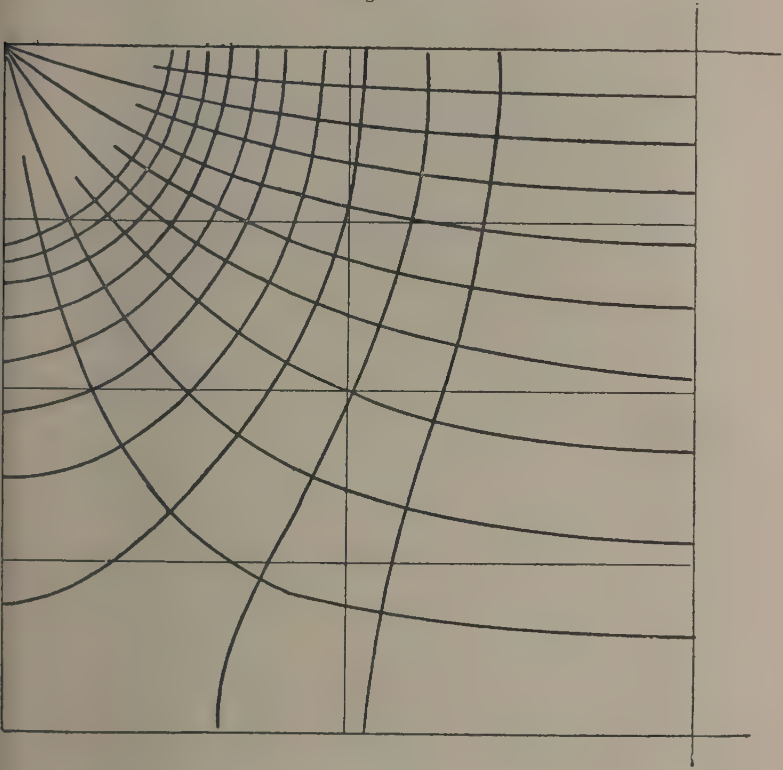
By this application of Lamé's formulæ, aided by those due to Hicks, we have discovered no new type of symmetry which allows two coordinates to be used instead of three. We have proved that within the stated limits the well-known types are the only possible ones. A summary of these may be useful.

*Summary of Types of Symmetry when the guiding lines are orthogonal to a family of surfaces.*

| Guiding lines. | Chequer ratio. | If $\nabla^2 V$ is made equal to $f(V, \alpha, \beta)$ over one surface $\gamma_0$ its value on the others will be | Analytical methods. |
|---|---|---|---|
| Parallel straights. | Constant. | $\nabla^2 V = f(V, \alpha, \beta)$. | Conjugate functions. |
| Circles with their centres on a common axis and their planes normal thereto. | Proportional to distance from axis. | $\nabla^2 V = f(V, \alpha, \beta)$. | Zonal harmonics of the cylindrical, spherical, spheroidal, and toroidal systems. |
| Radii from a common point. | Constant. | $\nabla^2 V = \dfrac{f(V, \alpha, \beta)}{r^2}$ where $r$ is the distance from the radiant point. | |

*Example of Symmetry about an Axis.*—Byerly in his 'Fourier's Series and Spherical Harmonics,' p. 230, sets the following problem:—"A cylinder of radius one metre and altitude one metre has its upper surface kept at temperature 100°, and its base and convex surface at the temperature 15°, until the stationary temperature is set up. Find the temperature at points on the axis 25 cm., 50 cm., and 75 cm. from the base. and also at a point 25 cm. from the base and 50 cm. from the axis." To solve this the first thing necessary is to prepare a chart bearing chequers of the appropriate shape for each distance from the axis. The graph of any solution of $\nabla^2 V = 0$ symmetrical about an axis would serve this purpose. For example several of the figures out of Maxwell's 'Electricity and Magnetism' would do. But I preferred to prepare a standard chart by ruling equidistant parallel equipotentials normal to the axis of revolution, and then stream-lines parallel to the axis at distances from it proportional to the square roots of the natural numbers 0, 1, 2, 3, 4, 5, &c. The cross section of the cylindrical shell enclosed between successive stream-lines is then the same for every pair, and the chequer ratio proportional to the distance from the axis. This having been done in red ink, a sheet of tracing-paper was pinned over it, the section of our given cylinder was drawn in black and equipotentials and lines of flow were drawn in pencil. These were then rubbed out and amended with the aim of making the pencil chequers everywhere very similar to the red rectangles underneath. When improvement became slow, the blurred lines were made firm and definite with ink and the chequers considered individually and marked as to whether they were too square or too thin. The lines were then drawn on a clean sheet of tracing-paper, the chequers again examined individually, and finally the lines fixed in ink (see fig. 3). Coordinate lines were then ruled and the values of V at their intersections were read from the graph. This process, from the ruling in of the given contour to the determination of V in numbers, took me four hours. The analytical method would perhaps have been more rapid *in this case*; but for an irregular shaped contour with an irregular boundary distribution the freehand solution would still take about the same time, while analytical methods may be almost indefinitely tedious.

Fig. 3.



The results are tabulated below :—

| | | | | |
|---|---|---|---|---|
| Distance from base of cylinder .... | 25 | 50 | 75 | 25 cms. |
| " axis " .... | 0 | 0 | 0 | 50 " |
| Potential read from the graph as decimal of unit range. | ·175 | ·38 | ·64 | ·14 |
| Last multiplied by the actual range (100°−15°) | 14·9 | 32·3 | 54·4 | 11·9 |
| Last +15° = actual temperature .. | 29·9 | 47·3 | 69·4 | 26·9 |

Not until these numbers had been written down did I look at the correct values found by Byerly from the Bessel function series, namely :—

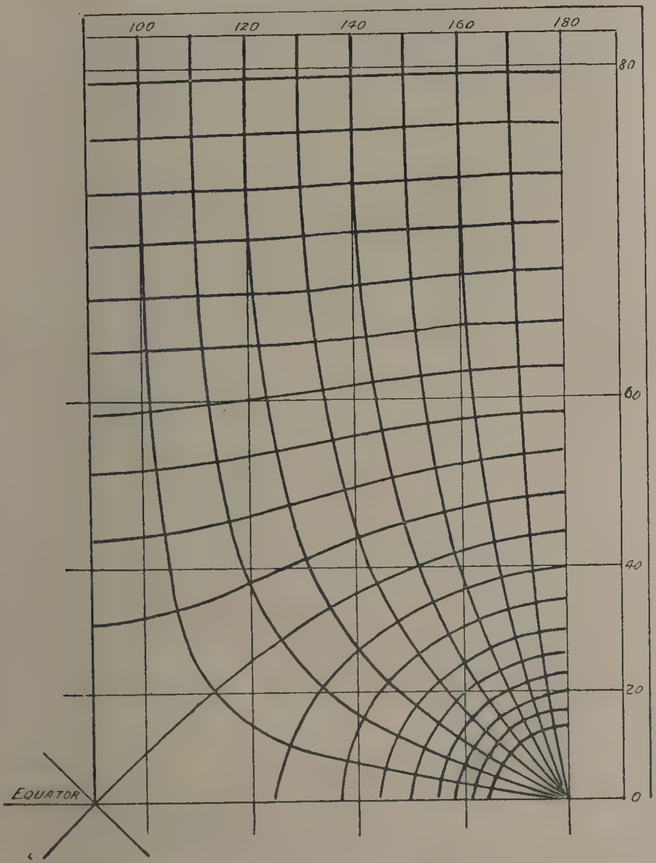
| | 29·6 | 47·6 | 71·2 | 25·8 |
|---|---|---|---|---|
| Errors of freehand determination.. | +·3 | −·3 | −·8 | +1·1 |

Disregarding signs the mean of these errors is ·63 which is equal to 0·74 per cent. of the range of 85°.

*Example of the type of symmetry when* V *is independent of the radius in spherical coordinates.*—“On a uniform spherical shell there are equal sources at the north and south poles and equal sinks at the extremities of a diameter lying in the equatorial plane. The sources and sinks send out and receive uniformly in all directions. The flux has no divergence except at the sources and sinks and no curl anywhere. Find the distribution of potential on the surface.” To do this we might draw orthogonal lines on the surface of a globe so as to make the chequers ratio constant. Or because, in Mercator’s projection, any small part on the globe transforms into a small part of the same shape on the map, we may transform the boundary conditions and obtain the required solution by drawing chequers of constant chequer ratio on the map. Blank Mercator projections suitable for this work may be obtained from George Philip & Son, Fleet Street. In the present example the lines of flow radiating from the pole become lines straight, parallel and equidistant at infinity. And as the graph progressed it was found that by their symmetry with the sinks on the equator, the foregoing condition must be very nearly satisfied at 10° from the poles, a region which is within the confines of the map. Again, in this case it is only necessary to determine V in one octant of the sphere, and symmetry helps us in other ways. The accompanying graph (fig. 4) is the best of four or five separate attempts. The time taken to make these was collectively four hours. Special attention was given to the equipotential curve which passes mid-way between the two equatorial sinks, and as the result of the aforesaid trials it was found to pass through a point 44° due north of the sink on the equator. This suggested that the true value should be 45°, and on looking at a sphere this is seen to follow from symmetry although it was not obvious on the map. Thus again we have a confirmation of the passable accuracy of the graphic method—the error here is 1 degree in 90 or 1·1 per cent. of the range.

So far we have only treated the problem as relating to a spherical shell. But we may next suppose the sphere solid and V to be independent of the radius. We will then have a solution of Laplace's equation in space. Since the chequer ratio is constant, the magnitude of the flux is inversely proportional to the linear dimensions of the chequer (on the

Fig. 4.



sphere not on the map) and is consequently proportional to $\frac{1}{r}$ along any guiding line, $r$ being the radius. But if we draw any small cone enclosing the polar axis—which is now a line source—we see that the outflow between two spheres $r$ and $r+\delta r$ is proportional to $\delta r \times$ (magnitude of flux) $\times$ (perimeter of the trace of the cone), and by the above this is proportional simply to $\delta r$. Therefore the polar axis must be a line source of uniform strength and similarly for any other source or sink when the guiding lines are straights passing through a common point—the strength must be independent of the radius.

*Case* (*b*).—General method for conduction in a *thin shell of any shape*, the thickness and conductivity being any given functions of position on its surface, and all conditions being constant throughout the thickness of the shell at any point of its surface. Take a solid bounded by a surface of the shape of the shell and draw small rectangles at numerous points of the surface, so that their chequer ratio $= \frac{\text{length along flow}}{\text{breadth across flow}}$ is directly proportional to the product of the thickness and conductivity at each point of the shell. For then the flow through each chequer will be the same. Suppose that these standard chequers are in some distinctive colour, say red. Now lay off in black the boundary conditions of the special problem and draw a black chequerwork to have the same chequer ratio as the red at each point, much as was done for symmetry about an axis. The standard red chequers need not be connected so as to form two systems of orthogonal lines but may be scattered anyhow over the surface, all that is necessary is that they should be sufficiently small and numerous.

Or it may be convenient to use a projection of the surface as was done in the case of the spherical shell above.

*Case* (*c*).—When there exists no family of surfaces normal to the guiding lines. Without pausing for generalities we will proceed at once to:—

*Screw symmetry about an axis.*—Let us discuss this with the aid of cylindrical coordinates $r, \phi, z$. At a point P on the axis OZ let a perpendicular be drawn extending to infinity. This perpendicular, which is to project only on one side of the axis, is imagined to revolve round the axis and slide along the same with proportional velocities. In one rotation round the axis let it move $l$ along the axis. Then the line sweeps out a surface, at all points of which the expression $z-\frac{l}{2\pi}\phi$ is constant. Let us put $z-\frac{l}{2\pi}\phi=\omega$.

Then as $\omega$ varies we pass from one of these screw surfaces to another formed by shifting the first parallel to $z$. The range of the coordinate $\omega$ is from 0 to $l$. The intersections of $\omega=$const. with the cylinders $r=$const. are a family of screw-threads.

Let $d\eta$ be an element of distance measured along any screw guiding line, so that $\frac{d\eta}{dz}=\frac{\text{length of turn of screw}}{l}=$ a function of $r$ only. And let us make V a linear function of distance along each screw-thread so that $\frac{\partial V}{\partial\eta}=$ a function of $r$ only.

Then as $d\eta$, $d\omega$, and $dr$ are in perpendicular directions they are independent and

$$\frac{\partial V}{\partial z}=\frac{\partial V}{\partial\omega}\cdot\frac{\partial\omega}{\partial z}+\frac{\partial V}{\partial\eta}\cdot\frac{\partial\eta}{\partial z}+\frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial z},$$

which reduces to $\frac{\partial V}{\partial z}=\frac{\partial V}{\partial\omega}$ + a function of $r$ only.

Therefore

$$\frac{\partial^2 V}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial V}{\partial\omega}\right)=\left(\frac{\partial\omega}{\partial z}\cdot\frac{\partial}{\partial\omega}+\frac{\partial\eta}{\partial z}\cdot\frac{\partial}{\partial\eta}\right)\frac{\partial V}{\partial\omega}$$

$$=\frac{\partial^2 V}{\partial\omega^2}+\frac{\partial\eta}{\partial z}\cdot\frac{\partial}{\partial\omega}\left(\frac{\partial V}{\partial\eta}\right)=\frac{\partial^2 V}{\partial\omega^2}.$$

Again,

$$\frac{\partial V}{\partial\phi}=\frac{\partial V}{\partial\omega}\cdot\frac{\partial\omega}{\partial\phi}+\frac{\partial V}{\partial\eta}\cdot\frac{\partial\eta}{\partial\phi}=\frac{l}{2\pi}\cdot\frac{\partial V}{\partial\omega}+\text{a function of } r \text{ only.}$$

Since

$$\frac{d\eta}{d\phi}=\frac{\text{length of turn of screw}}{2\pi}=\text{a function of } r \text{ only.}$$

And so

$$\frac{\partial^2 V}{\partial\phi^2}=\frac{l}{2\pi}\cdot\frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial\omega}\right)=\frac{l}{2\pi}\left(\frac{\partial\omega}{\partial\phi}\cdot\frac{\partial}{\partial\omega}+\frac{\partial\eta}{\partial\phi}\cdot\frac{\partial}{\partial\eta}\right)\frac{\partial V}{\partial\omega}$$
$$=+\frac{l^2}{4\pi^2}\cdot\frac{\partial^2 V}{\partial\omega^2}.$$

Now substitute these values of $\frac{\partial^2 V}{\partial z^2}$ and $\frac{\partial^2 V}{\partial\phi^2}$ in the expression for $\nabla^2 V$ in cylindrical coordinates, and we have

$$\nabla^2 V=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\left(1+\frac{l^2}{4\pi^2 r^2}\right)\frac{\partial^2 V}{\partial\omega^2}$$

which contains only two coordinates $r$ and $\omega$. So that if we make $\nabla^2 V=f(V, r, \omega)$ over any surface the same will be true throughout the whole region filled by the screw-threads passing through the surface, provided that $\frac{\partial V}{\partial\eta}$ is such as to make $\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}$ constant along every guiding screw. One way of satisfying this is to make V increase by the same amount per turn of the screw, along every screw-thread and so that $\frac{\partial^2 V}{\partial r^2}$ and $\frac{\partial V}{\partial r}$ are both constant along every guiding screw.

In the following pages, except where specially indicated, we will consider only the case $\frac{\partial V}{\partial\eta}=0$.

As this result does not appear to be given in the text-books, it may be well to confirm it by a slightly different line of reasoning, as follows. The tangent of the angle between the tangent to any screw-threads and a plane normal to the axis of symmetry is $\frac{l}{2\pi r}$.

Therefore the first space rate of any function of position V along the tangent to a screw-thread is

$$\frac{1}{\sqrt{l^2+4\pi^2 r^2}}\left\{2\pi r\frac{\partial}{r\partial\phi}+l\frac{\partial}{\partial z}\right\}V.$$

If $\frac{\partial V}{\partial\eta}=$ a function of $r$ only, along every screw-thread then we have a function of $r$ only $+2\pi\frac{\partial V}{\partial\phi}=-l\frac{\partial V}{\partial z}$

throughout the whole region. Therefore this last equation will still remain true after differentiation by $\phi$ or by $z$, thus

$$-\frac{2\pi}{l}\frac{\partial^2 V}{\partial\phi^2}=\frac{\partial^2 V}{\partial\phi\partial z},$$

$$\frac{\partial^2 V}{\partial\phi\partial z}=-\frac{l}{2\pi}\frac{\partial^2 V}{\partial z^2}.$$

Equating the two values of $\dfrac{\partial^2 V}{\partial\phi\partial z}$ thus obtained we have

$$\frac{\partial^2 V}{\partial\phi^2}=\frac{l^2}{4\pi^2}\frac{\partial^2 V}{\partial z^2},$$

which on substitution in the expression for $\nabla^2 V$ gives

$$\nabla^2 V=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\left(1+\frac{l^2}{4\pi^2 r^2}\right)\frac{\partial^2 V}{\partial z^2}.$$

But it is now to be observed that if the distribution of V on any plane passing through the axis of symmetry is known, then V is determined everywhere. And on such a fixed plane the contours of $z$ are identical with those of $\omega$. So that we may replace $z$ by $\omega$ in the last equation, and the previous result is confirmed.

We have shown that if we make $\nabla^2 V=f(V, r, \omega)$ over any surface intersecting all the screw-threads, the same will be true throughout the whole region, with the stated provisions as to the value of $\dfrac{\partial V}{\partial\eta}$. The geometrical meaning of this result is that if we draw any infinitesimal rectangle normal to one of the screw-threads and draw screw-threads through each of its four corners, then the infinitesimal tube thus formed will be everywhere rectangular in normal cross section, and more than this, the rectangle will have the same ratio of length to breadth and will be of the same size at all points along the tube. For if we consider one pair of opposite faces of the tube as equipotentials and the other pair as lines of flow, then these properties are seen to follow from the fact that $\nabla^2 V$ is constant along a screw-thread when V is constant along the same. And indeed these properties are immediately obvious from the appearance of the system.

Consequently, if we take any family of surfaces $\alpha$ passing through the guiding screw-threads, there will always be an orthogonal family of surfaces $\beta$, also passing through the screw-threads. If the surfaces $\alpha$ are the contours drawn at equal intervals of the potential V the surfaces $\beta$ are stream-surfaces. And $\frac{H_\alpha}{H_\beta}$ may be named the "Chequer Ratio" consistently with what has gone before. If $\nabla^2\alpha$ is to vanish we must have

$$\frac{\partial}{\partial\alpha}\left(\frac{H_\alpha}{H_\beta \times \begin{pmatrix}\text{length of portions of successive screw-threads intercepted}\\ \text{between two stream-lines lying on the same stream-surface}\end{pmatrix}}\right)=0.$$

Since the screw system is uniform the length of the portions of successive screw-threads intercepted between two stream-lines lying on the same surface $\beta$ can be proportional to nothing else than the length of one turn of the screw-thread at the radius considered. For the two stream-lines in question must by symmetry make equal angles with planes normal to the axis of the screw, at each pair of points lying on the same screw-thread. So that the projection of the distance between the said pair of points onto the axis of the screw will be always the same fraction of $l$ as the points move from one screw-thread to another.

Now the length of an arc $ds$ of a screw-thread being

$$\sqrt{dz^2+r^2d\theta^2}=d\theta\sqrt{\frac{l^2}{4\pi^2}+r^2},$$

the length of one turn is

$$\int_{\theta=0}^{\theta=2\pi} ds=2\pi\sqrt{\frac{l^2}{4\pi^2}+r^2}=l\sqrt{1+\frac{4\pi^2r^2}{l^2}}.$$

Therefore we may satisfy the equation $\nabla^2\alpha=0$ by making $\frac{H_\alpha}{H_\beta}$ proportional to $\sqrt{1+\frac{4\pi^2r^2}{l^2}}$ as the radius varies. A table giving the values of $\sqrt{1+\frac{4\pi^2r^2}{l^2}}$ for a number of values of $\frac{r}{l}$ is annexed.

| $\frac{r}{l}$. | $\sqrt{1+\frac{4\pi^2r^2}{l^2}}$ $l$ times this equals arc of one turn of thread. | For standard chart. | | $\frac{l}{2\pi r}$ equals tangent of angle between helix and circle. |
|---|---|---|---|---|
| | | $H_\alpha$. | $H_\beta$. | |
| 0 | 1·0000 | ·5507 | 1·8157 | ∞ |
| ·05 | 1·0482 | ·5639 | 1·7735 | 3·1831 |
| ·10 | 1·1811 | ·5985 | 1·6707 | 1·5916 |
| ·15 | 1·3741 | ·6456 | 1·5490 | 1·0610 |
| ·20 | 1·6060 | ·6979 | 1·4328 | ·7958 |
| ·30 | 2·1338 | ·8045 | 1·2430 | ·53052 |
| ·40 | 2·7049 | ·9058 | 1·1040 | ·39789 |
| ·50 | 3·2969 | 1·0000 | 1·0000 | ·31831 |
| ·60 | 3·9003 | 1·0877 | ·9194 | ·26526 |
| ·70 | 4·5105 | 1·1697 | ·8550 | ·22736 |
| ·80 | 5·1251 | 1·2468 | ·8021 | ·19894 |
| ·90 | 5·7426 | 1·3198 | ·7577 | ·17684 |
| 1·00 | 6·3623 | 1·3892 | ·7199 | ·15916 |
| 1·5 | 9·478 | 1·6955 | ·5898 | ·10610 |
| 2·0 | 12·606 | 1·9554 | ·5114 | ·07958 |
| 2·5 | 15·740 | 2·1850 | ·4577 | ·06366 |
| 3·0 | 18·876 | 2·3928 | ·4179 | ·05305 |
| 3·5 | 22·014 | 2·5840 | ·3870 | ·04547 |
| ∞ | ∞ | ∞ | 0 | ∞ |

It is hoped that these values are correct to less than half a unit in the last place.

As there is no surface normal to the screw-threads, it is not possible to draw standard rectangles of the appropriate chequer ratio for each distance from the axis. But as the whole distribution of V is determined when the section of it by a plane passing through the axis of the screw is known, we may draw on this plane the sections of tubes formed by the surfaces $\alpha$ and $\beta$ in such a way that $\frac{H_\alpha}{H_\beta}$ is proportional to

$$\sqrt{1+\frac{4\pi^2r^2}{l^2}}.$$

The sections of these tubes will in general not be rectangles; in fact, the angles and ratio of sides of the chequers formed by the traces of $\alpha$ and $\beta$ on the plane $\phi=$constant will both now depend on the orientation of the chequer as well as on its distance from the axis. It will therefore be necessary to make a chart of standard chequers in various orientations at a number of distances from the axis. Plate V. is such a chart. The rectangles in the right-hand margin represent normal cross sections of the tubes formed by the surfaces $\alpha$ and $\beta$. In a line with each of these are five sections of a tube of the same size and shape by the plane of the paper, when the angle between one face of the tube and the normal to the axis of the screw is successively 0°, 22½°, 45°, 67½°, 90°. In order to be clearly visible these parallelograms are drawn quite large. What each really represents is the shape of an infinitesimal chequer situated at the central point of the large one. Practically the difference will not be important.

Now this standard diagram can be covered by a sheet of tracing-paper, and two intersecting families of lines drawn on the tracing-paper in such a way that the parallelograms formed by them are everywhere similar to the chequers underneath, which have the same distance from the axis and the same orientation on the paper. Then if this tracing-paper plane rotate round the axis and slide along it so as to follow the guiding lines, the equipotential lines on the paper will sweep out the contours at equal intervals of V in space in such a way that $\nabla^2 V = 0$ and the other family of lines will sweep out stream-surfaces.

A quantity which it is frequently necessary to determine is the magnitude of the flux

$$=\sqrt{\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2}=\frac{1}{H_\alpha}.$$

Since
$$\frac{H_\alpha}{H_\beta}=A\sqrt{1+\frac{4\pi^2r^2}{l^2}},$$

where A is an absolute constant, we must have

$$\frac{1}{H_\alpha}=A\frac{\sqrt{1+\frac{4\pi^2r^2}{l^2}}}{H_\beta}$$

So that $\beta$ is a stream function analogous to the forms in use when the guiding lines are parallel straights or circles with their centres on, and their planes normal to, a common axis.

In types previously studied, when the graph was drawn on a surface normal to the guiding lines, $H_\alpha$ and $H_\beta$ were proportional to the length and breadth of a chequer and could be measured directly. But here we must first compare the linear dimensions of a freehand chequer with those of the standard oblique section of the tube bounded by two stream-surfaces and two equipotentials, and then refer to the normal section of the same tube in the right-hand margin of the chart.

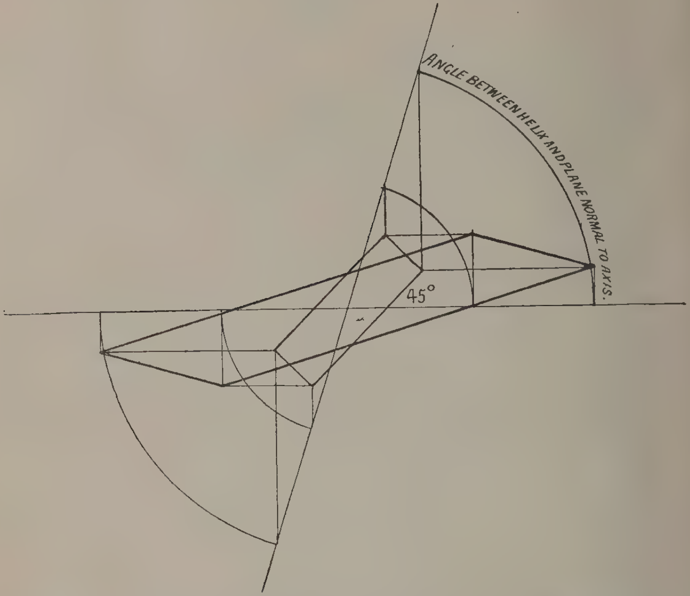
The standard chequers were obtained in the following manner:—

$\frac{H_\alpha}{H_\beta}$ being equal to constant $\times \sqrt{1+\frac{4\pi^2 r^2}{l^2}}$, some other relation is necessary to determine $H_\alpha$ and $H_\beta$ separately. The relation $H_\alpha \times H_\beta = 1$ was chosen for this purpose, as this gives a neat appearance to the standard chart. It was also found convenient to make the constant such that $\frac{H_\beta}{H_\alpha} = 1$ when $\frac{r}{l} = 0{\cdot}5$. The values of $H_\alpha$ and $H_\beta$ were calculated and are given in the accompanying table. The sides of the rectangles in the right-hand margin of the standard chart were drawn proportional to $2H_\alpha$ and $2H_\beta$.

To obtain the slant section, the tangent of the angle between the tangent a guiding-line and the plane normal to the axis of the screw, was first calculated. It is equal to $\frac{l}{2\pi r}$, and is given in the table under that head. The rectangles were then projected with ruler and compasses in a manner which is perhaps sufficiently indicated by fig. 5, which shows the construction when $\frac{r}{l} = 0{\cdot}05$ and the angle between a radius from the axis of symmetry and the tangent plane to the surface $\alpha$=constant meeting at the point considered is 45°.

Of the innumerable solutions of $\nabla^2 V = 0$ possessing screw symmetry of the sort described, which may be obtained by the aid of this standard chart, perhaps the simplest is the field due to a helical line source, such for example as the distribution of temperature in a mass of electrically insulating

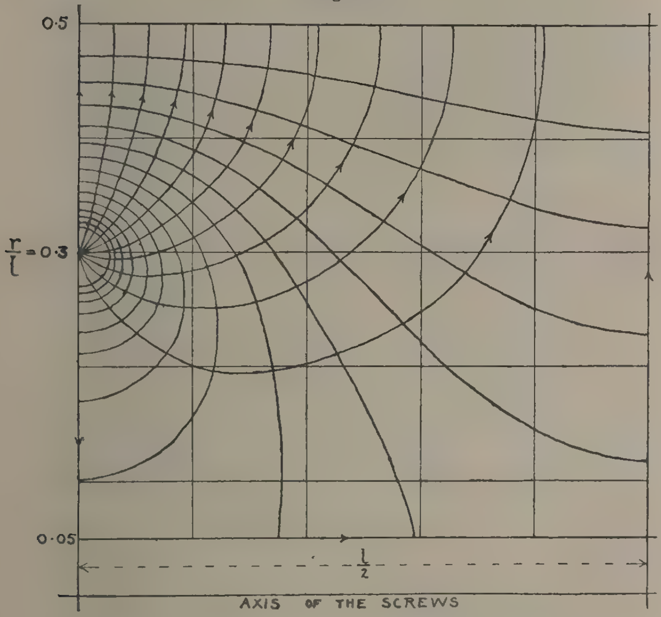
Fig. 5.



material which encloses a helical copper wire carrying an electric current. To avoid the introduction of a difficulty not characteristic of screw symmetry, I have assumed a core of non-conducting material in the form of a circular cylinder surrounding the axis. This relieves us of the necessity of considering the axial line of equilibrium, which would otherwise have to be treated by an extension of the method in Section IV. The external surface of the medium is also taken as a circular cylinder and is assumed to be at constant temperature. Consistently with our boundary conditions we may suppose that $\frac{dV}{d\eta}=0$. Now symmetry will help us in several ways, for since the chequer ratio on the standard chart is the same whether any particular half-turn of the screw passes over or under the chart, one sees on beginning to make the drawing, that the two surfaces $\omega=$ const., which pass respectively through the electric current and half-way between two adjacent turns of the current, must be surfaces of flow. Again, very close to the electric current the flow of heat will be nearly the same as that due to a straight current tangential to the helix, that is to say, the lines of flow will be normals to the helix and the isothermals will approximate to circular cylinders concentric about the tangent.

The particular dimensions chosen were $\frac{r}{l}=0{\cdot}05$ for the core, 0·3 for the source, and 0·5 for the outer cylinder.

Fig. 6.



Owing to orientation of the chequer affecting its shape this graph took twice or thrice as long to adjust as did the others in this paper. Its errors are discussed in Section VIII. hereafter.

The magnetic field due to the helical current may doubtless be determined in a very similar manner. Here $\frac{\partial V}{\partial \eta}$ will be a constant other than zero and the cyclical properties of the field will add a further complication.

## IV.

We have hitherto passed over without mention the peculiarities relating to points of equilibrium—these are *points at which the first space-rate of the potential vanishes in all directions.* In the neighbourhood of these the chequers become unusually large, and if any chequer goes right up to an equilibrium point it will not have the shape characteristic of its neighbours, but will take a peculiar form of its own.

There are several diagrams of this in Maxwell's 'Electricity and Magnetism.' See, for example, vol. ii. fig. xvii.

Now if V be expressed in terms of rectangular coordinates $u$ and $v$ lying in the plane of the graph with their origin at the equilibrium point, then linear terms in V must vanish, and we have

$$V=Au^2+B\omega+Co^2+Eu^3+Fu^2v+Guv^2+\text{ terms of higher degree.}$$

Now let us make $\nabla^2V$ vanish.

For guiding lines parallel straight and normal to the plane of the graph

$$\nabla^2V=\frac{\partial^2V}{\partial u^2}+\frac{\partial^2V}{\partial v^2}=2(A+C)+u(6E+2G)+v(2F+6H).$$

When the graph is on a plane passing through an axis about which there is symmetry of revolution and $u$ is normal to this axis, we must add to the above value of $\nabla^2V$ the term

$$\frac{1}{r}\frac{\partial V}{\partial u}=2\frac{u}{r}A+\frac{v}{r}B+3\frac{u^2}{r}E,$$

where $r$ is distance from the axis.

Now when the point considered is not on or close to the axis, it will be possible to put in so many chequers that the first two chequers in any direction from the equilibrium point require for their measurement so small a range of $u$ and $v$ that the fractions $\frac{u}{r}, \frac{v}{r}$, &c., will be small, and therefore the additional terms which come in for symmetry about an axis may be neglected, and we have the same form for $\nabla^2V$ in both cases.

Further, since Mercator's projection does not alter the shape of any small pieces, $\nabla^2 V$ will have the same form in the neighbourhood of an equilibrium point on the Mercator's plan of the distribution on a sphere.

This being so, the general form of V in all three cases is

$$V = a(u^2 - v^2) + b \,.\, uv + g(u^3 - 3uv^2) + h(v^3 - 3u^2 v) + \text{higher terms},$$

where $a$, $b$, $g$, and $h$ are arbitrary constants. When the ratios of $a$ and $b$ to the succeeding coefficients do not vanish, then the first two terms are all that we need consider. Now it may easily be shown that by a proper rotation of the axes of reference, so that $u\,v$ tranform to $u_1 v_1$, the sum of these two terms may be transformed into either of them separately We need therefore only consider one, say $bu_1 v_1$. The contours of this function are hyperbolas and are orthogonal to those of $a(u_1{}^2 - v_1{}^2)$. The ratio $\frac{a}{b}$ is determined by the chequer ratio in the neighbourhood of the equilibrium point.

A graph of this function for the special case of unit chequer ratio is given in Webster's 'Dynamics,' p. 525, and shows that two equipotentials meet at right angles at the equilibrium point, and that two stream-lines also pass through the same point and bisect the angles between the equipotentials. The eight curved chequers which meet in the point each have consequently three corners of 90° and one of 45°. A graph of this function may be used as a "standard equilibrium point" to keep the eye informed of the necessary proportions of the first and second ring of chequers surrounding the point.

If, however, the coefficients $a$ and $b$ vanish, while $g$ and $h$ do not, then the terms of the 3rd degree become all important.

By rotating the axes the sum of the two terms of the 3rd degree may be reduced to either separately. A rough graph of the contours of these functions is given by Fiske in Merriman & Woodward's 'Higher Mathematics,' p. 248. Here three equipotentials intersect in the equilibrium point. And three stream-lines bisect the angles of 60° which are formed in this way.

Now when a graph has to be drawn and is found to contain an equilibrium point, the general arrangement of the potential will give us the clue as to whether two, three, or more equipotentials intersect in the equilibrium point. And this being known, we have only to draw in the standard type at the proper dimensions and chequer ratio.

When the graph is drawn on a plane passing through an axis about which there is screw symmetry of the sort described in Section III *c*, then the appearance is different, for we have to add to the value of $\nabla^2 V$ for circular symmetry about an axis the term

$$\frac{l^2}{4\pi^2 r^2}\cdot\frac{\partial^2 V}{\partial v^2}=\frac{l^2}{4\pi^2 r^2}(2C+2uG+6vH).$$

And therefore writing

$$1+\frac{l^2}{4\pi^2 r^2}=R^2$$

we have

$$V=a(R^2u^2-v^2)+b\,.\,uv+g(R^2u^3-3uv^2)+h(v^3-3R^2u^2v).$$

A simpler way of looking at the matter is to consider a tiny plane element normal to the guiding screw which forms the line of equilibrium. The normals to the surfaces $\omega=$const. lie in this plane. If $dS_\omega$ distance along such a normal, then

$$d\omega=dS_\omega\sqrt{\left(\frac{\partial\omega}{\partial z}\right)^2+\left(\frac{1}{r}\frac{\partial\omega}{\partial\phi}\right)^2}=dS_\omega\sqrt{1+\frac{l^2}{4\pi^2 r^2}}.$$

Substituting this in the expression of $\nabla^2 V$ in terms of $r$ and $\omega$ we have

$$\nabla^2 V=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial S_\omega{}^2},$$

just as if $S_\omega$ was $z$ in circular symmetry about an axis. From this we see that the appearance of the equilibrium point on a small plane element normal to the guiding screw will be exactly similar to the forms already dealt with. Its appearance on a plane which passes through the axis of the screw may be sketched without much difficulty by comparing the chequers in the right-hand margin of the standard chart with their projections as drawn in the middle of the chart.

### V. *Equations other than Laplace's.*

It has been shown above that in order to solve the equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\text{any given function of } V,\ x,\ y,$$

a relation between differences of chequer ratios has to be satisfied. And the same will be found to be true for the other forms of the equation $\nabla^2 V =$ a function (of V and of position) which can be treated by two coordinates. A difference relation of the sort referred to would involve the comparison of each chequer with a standard set having graded chequer ratios, followed by the calculation of $\nabla^2 V$ by arithmetic. And although it would doubtless be possible to carry out the necessary operations, yet it would almost certainly be quicker and more accurate to use arithmetical finite differences altogether, writing in the numerical values of V at a set of points on the paper and adjusting these numbers until the finite difference equation is satisfied,—in a manner which may be described in a future paper. In view of this I will not attempt to elaborate freehand methods for $\nabla^2 V =$ a given function of V and of position.

There are, however, certain common space distributions which may be treated graphically with simplicity although they do not satisfy $\nabla^2 V = 0$.

Firstly, *when the conductivity is a continuous function of position*, and the direction of the flux is normal to the contours of a potential, and the magnitude of the flux is the maximum space-rate of the potential multiplied by the conductivity, and the flux has no divergence. For example: the flow of heat and electricity in isotropic but non-homogeneous bodies, or the soakage of water in a saturated subsoil the upper layers of which are more porous than those below. Let K be the conductivity and suppose that it is constant along each guiding line but varies from one such line to another. Then, when the lines have a family of surfaces normal to them we must have

$$\frac{\partial}{\partial \alpha}\left(\frac{H_\beta \,.\, H_\gamma \,.\, K}{H_\alpha}\right)=0$$

in order that the flux shall not diverge. This is very easily assured by preparing the paper with standard chequers having their chequer ratio $\frac{H_\alpha}{H_\beta}$ proportional to $H_\gamma \, . \, K$. In fact, we have an example of this in Section III *a* above; for circular symmetry about an axis may be regarded for this purpose as flow between parallel planes in a medium having conductivity directly proportional to the distance from the axis. And reciprocally.

Similarly in the case of screw symmetry, standard chequers are to be prepared having

$$\frac{H_\alpha}{H_\beta} \text{ proportional to } K\sqrt{1+\frac{4\pi^2r^2}{l^2}}.$$

Two other cases can probably be treated freehand, namely, the *flow of heat in bodies where the conductivity varies with the temperature*, and, of great practical importance, *the distribution of magnetic induction in soft iron, taking into account the variation of the permeability with the force*. But these again will be left to those who need the results.

## VI. *Note on Boundary Conditions.*

It may be convenient to the reader if we bring together certain well-known facts concerning boundary conditions.

Let us regard V simply as a function of position, not necessarily satisfying $\nabla^2 V=0$ or any other equation; and, as always, let contours be drawn at small intervals of V each equal to $k$. Then the first space-rate of V in any direction at a point is inversely as the intercept cut off from a line in that direction by two contours of V one on each side of the point, and is directly as K. Suppose, further, that the whole distribution of V can be represented by a single graph.

1. If we have to make V such that the magnitude and direction of its maximum first space-rate, the Hamiltonian vector $\nabla V$, satisfies given values over a boundary of a given shape. Then it is easy to set off the ends of the contours of V with a ruler and scale, for their directions are known and also the distance apart of successive pairs.

2. If we are not given $\nabla V$ over the boundary but only the first space-rate of V in a given direction. Then there are an indefinite number of ways in which the contours of V may cut the boundary ; and as it will not generally be possible to say which of these is consistent with the internal conditions, they must be drawn and modified freehand as the approximation to the internal conditions proceeds. This is usually not difficult.

3. To *make* V *continuous* at any surface cutting the distribution, all that is necessary is that the ends of the contours of V approaching from the two sides should meet one another at this surface. Whether they meet at an angle or not does not matter.

4. To *make the first space-rates of* V *in every direction continuous* at any surface where V is continuous, not only must the contours of V meet one another, but they must pass smoothly into one another without making an angle. For if they made an angle and a straight line were drawn tangent to one branch of the contours at the angle, then the ratio of successive intercepts of this line by the contours of V would not become unity when the contours were drawn at indefinitely small intervals of V, so that the second space-rate along this straight would be indefinite at the angle.

5. Suppose next that a non-divergent vector is normal to the surfaces $V = \text{const.}$, and that the magnitude of the vector is equal to the space-rate of V along the said normal, multiplied by a scalar function of position ; which according to the particular application will be the conductivity, permeability or some other specific constant. Then we may require the conditions which must hold at a *boundary where the specific constant has a discontinuity* while V is continuous—as, for example, where magnetic flux passes from air into mild steel. These conditions, which I take from Prof. J. J. Thomson's ' Elements of Electricity and Magnetism,' may be stated thus :—If $K_1$, $K_2$ are the aforesaid specific constants on the two sides of the boundary, and $\theta_1$ and $\theta_2$ are the corresponding angles which the direction of the vector makes

with the bounding surface, then

$$\frac{1}{K_1}\tan\theta_1 = \frac{1}{K_2}\tan\theta_2.$$

Now if the graph be drawn on a surface which is normal to the guiding lines, the direction of the vector lies in the graph, and $\theta_1$ and $\theta_2$ are the actual angles which one sees. The same is true of the Mercator's map of a spherical surface distribution, since the angles are unchanged by projection. But with our method for screw symmetry, the angles $\theta_1$ and $\theta_2$ do not immediately appear, and comparison must be made with the angles of the slant sections of the rectangular tubes given on the standard chart.

## VII. *Miscellaneous Notes on Draughtsmanship.*

(*a*) Since with the exception of given boundaries and lines deduced from symmetry no part of the field can be said to be correct until the whole field is correct, it is advisable to begin by covering the whole field with intersecting lines, however erroneous they may be, and then to carry out amendments over wide areas at one time.

(*b*) In the final stages of a drawing intended improvements often overshoot the mark or cause unforeseen disturbances in the surrounding chequers. It seems well, therefore, to lay aside the indiarubber after a certain accuracy has been reached, and, placing a sheet of tracing-paper over the rough diagram, to draw the intended improvements upon this. And so with all later stages. The tracing-paper diagrams are then compared with one another and the best selected.

(*c*) The graphic addition of two scalar functions of position is conveniently performed in the way described by Maxwell (‘ Elementary Treatise on Electricity ’) by laying the contours drawn on a sheet of tracing-paper at equal intervals of the one over those of the other, covering the two with a clean sheet of tracing-paper, and drawing the diagonals of the chequers formed by the intersecting contours.

## VIII. *Estimation of Errors.*

To one reading an account of this freehand method without having worked an example, it might seem as if there were no way of setting a limit to the errors of any particular graph. This, if it were true, would be a serious fault. But, happily, it is not so; for it is commonly necessary to make several drawings and then select the best of them: so that by the time the draughtsman has reached a drawing which he can scarcely improve upon, he has before him deviations from it in divers directions. The difference, then, between the selected graph and the second best graphs is a measure of the errors of the latter and an outside limit to the errors of the former. The actual errors of the selected graph will be less than this limit, and may be estimated by comparing the errors in the shape of the individual chequers in the best and second best graphs, and taking a fraction, thus:—

$$\frac{\text{individual chequer error in best graph}}{\text{same in second best graph}}\left(\begin{matrix}\text{difference between best and}\\ \text{second best graphs}\end{matrix}\right).$$

This is the true measure of the errors of the best graph. It depends, of course, on a general mental estimate or appreciation, and is consequently not susceptible of exact definition. But this does not much matter, for if the value of an error be known within two times either way it is usually sufficient.

The difference between the best and second best graphs is less dependent on a mental estimate, and consequently sets a firmer limit to the possible error.

Taking, for example, the graph of the field round a helical line source given in section III. *c*, and laying over it the tracing of the unpublished second-best graph, one sees that the difference in position of the lines in the two graphs nowhere exceeds $\frac{1}{2}$ the linear dimension of the chequer, at the point and in the direction considered. Now I should estimate that the error of the shape of individual chequers in the published graph averaged $\frac{1}{2}$ of the same quantity in the other; so that $\frac{1}{4}$ of the linear dimensions of the chequer may be taken as the error of position of the lines in the published graph. Now the graph exhibits ten tubes of flow; so that $\frac{1}{4}$ of one tube is $2\frac{1}{2}$ per cent of the range. This is in the worst parts of the field. Elsewhere the error will be less, but it may still be expected to exceed the errors found when the graph is drawn on a surface normal to the guiding lines, because in the case of screw symmetry we have the added difficulty that the shape of the chequers depend upon its orientation.

## DISCUSSION.

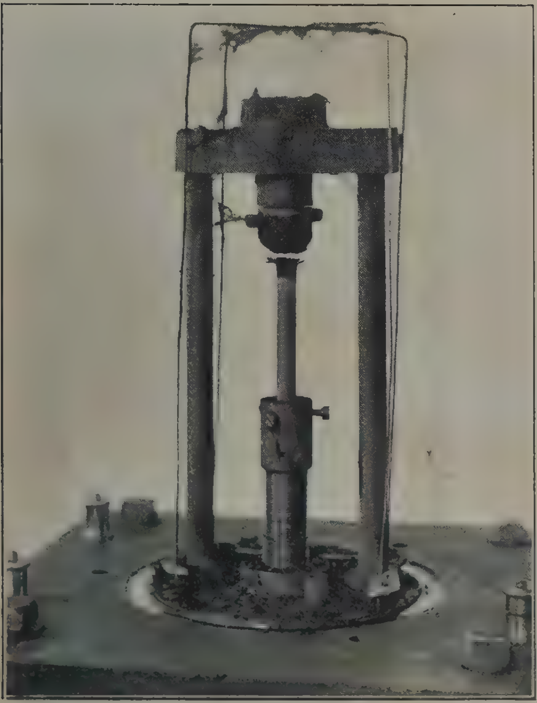
Mr. A. RUSSELL said that electricians had often to draw lines of force freehand. As a rule they had little more than their intuition to guide them, as the values of many of the variables were only known roughly and the bounding-curves were not simple geometrical figures. They would therefore welcome the paper, as it would enable them to improve their diagrams. The problem of the flow in two-dimensional space could in many cases be satisfactorily solved by experiment—for instance, by Prof. Hele Shaw's method. The author had shown, however, that the graphical method could be applied usefully to certain classes of problems in three dimensions which it would be extremely difficult to attack by any other method. The speaker had found difficulty in determining the "chequer ratio" in electrical problems where the extent of the field was unlimited.

Mr. A. CAMPBELL asked if the author's method would be applicable to a helix of finite length carrying a current.

The CHAIRMAN remarked that in the method described much depended on the eye, judgment, and patience.

The AUTHOR, in reply to Mr. Campbell, said his method would not apply to the case of a helix of infinite length.

FIG. 4.



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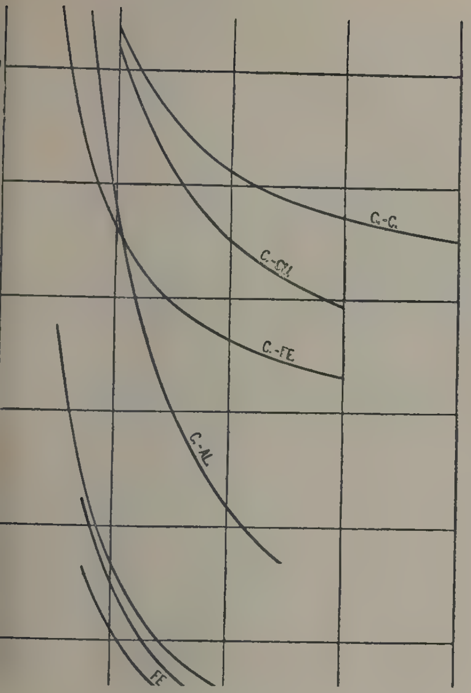
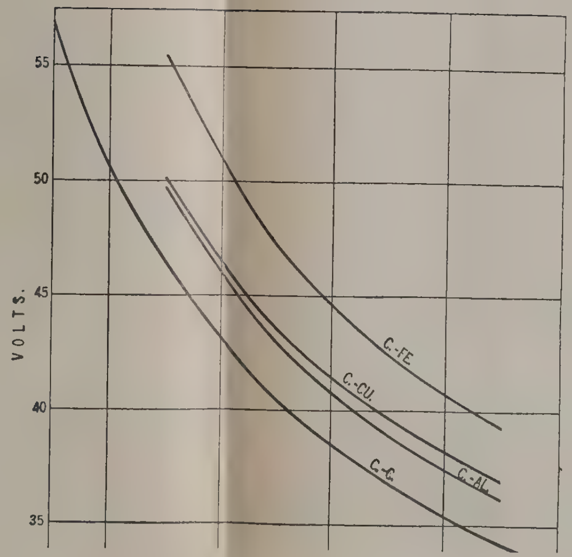


Fig. 6.



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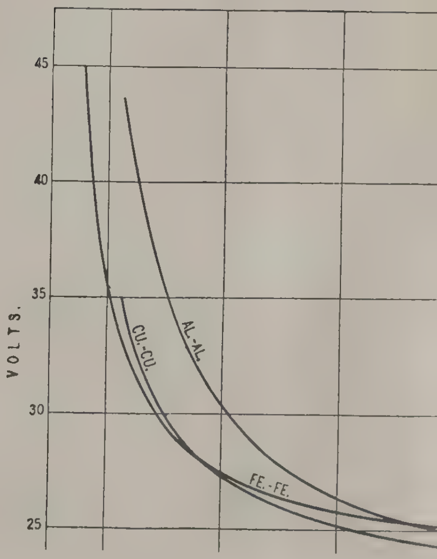

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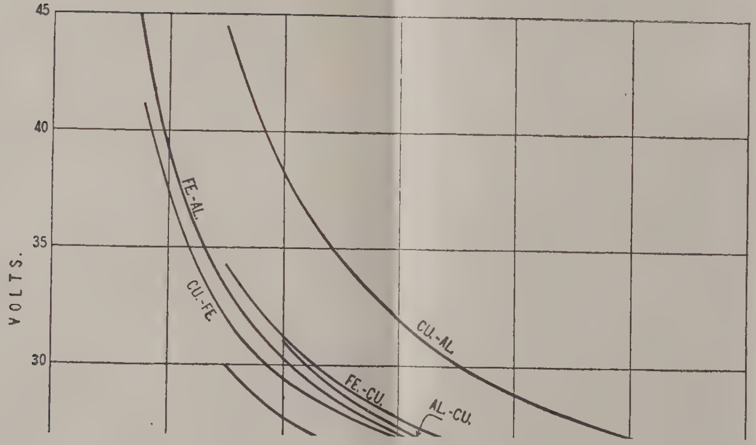
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Fig. 9.

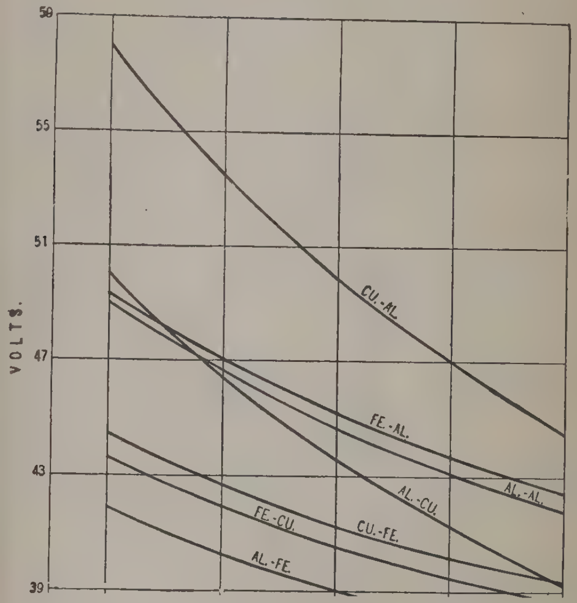
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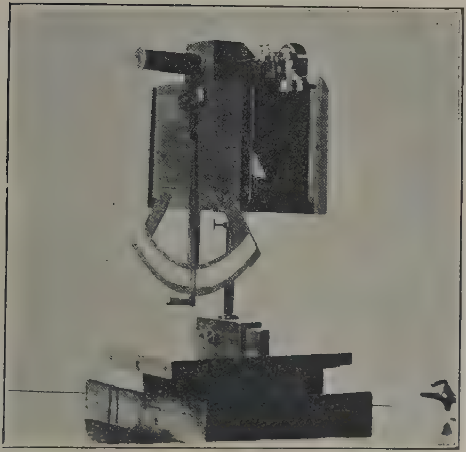
Fig. 1.



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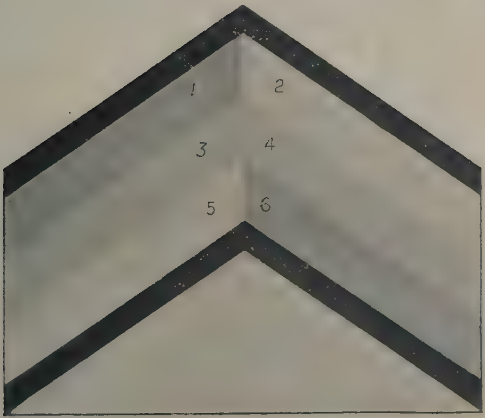

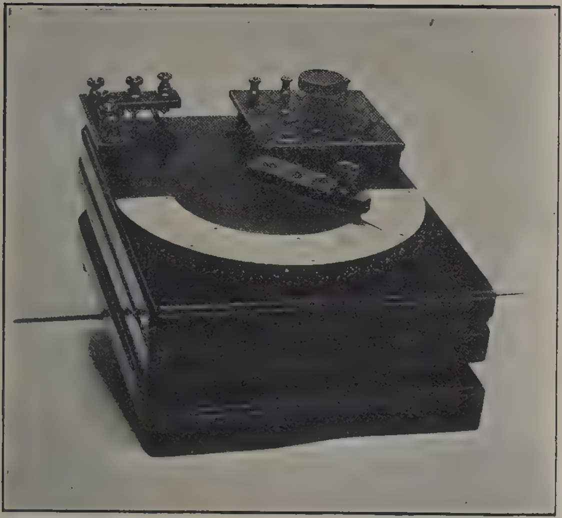

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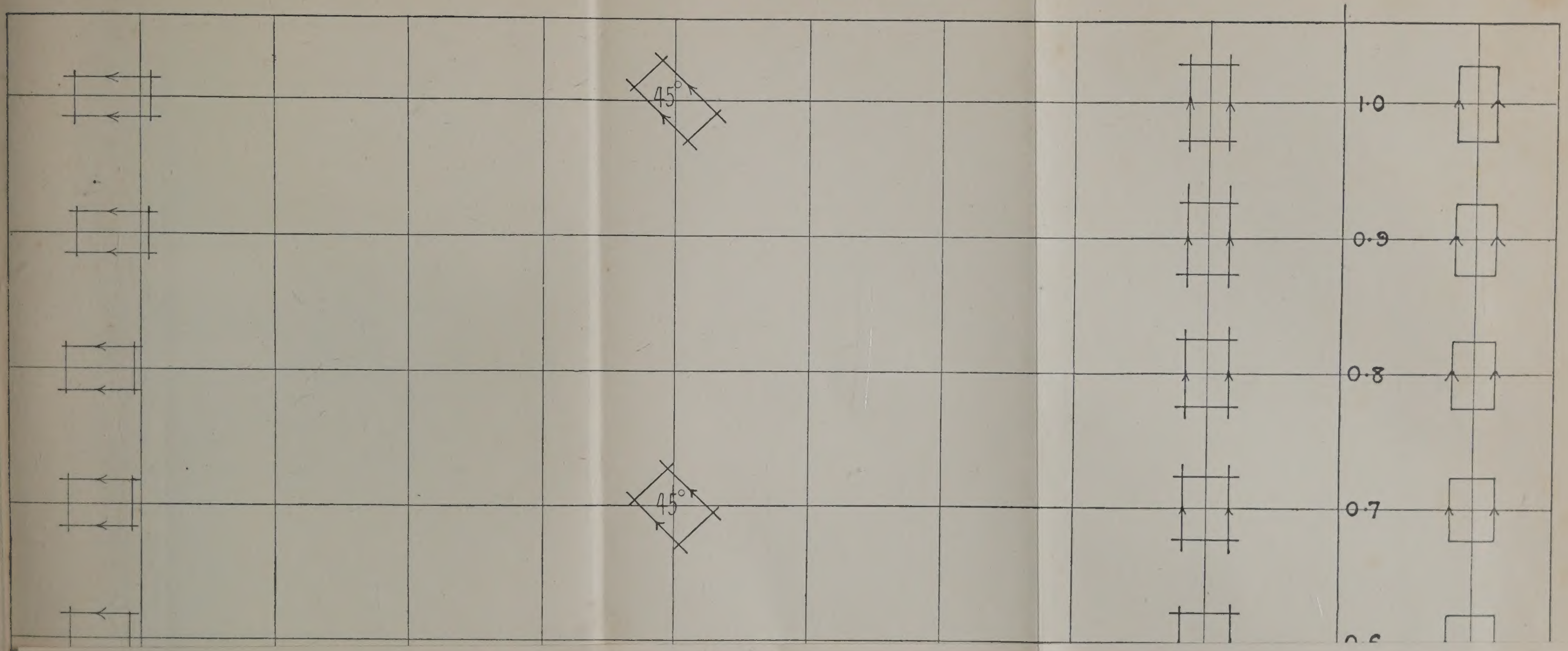

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